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Bipan Hazarika · Santanu Acharjee ·
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Editors

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Preface

This book covers the key developments in dynamical systems and topology, as well as their multidisciplinary applications in a variety of fields, including computer science, cosmology, nanofluids, etc. Establishing a link between dynamical systems and new advancements in multidisciplinary topology study is its primary goal. Additionally, this book provides solutions for researchers, specialists, etc. who are looking for cutting-edge research trends in various fields. It also creates opportunities for the use of topology and dynamical systems to solve interdisciplinary problems in different areas. The chapters highlight developments in dynamical systems and topology as well as their applications to real-world issues and other areas of mathematics. The authors of the chapters are experts in these two fields.

The joint operations of * and Ψ in the ideal topological space are covered in the first chapter "Closure Functions Induced by $*$ and Ψ Operators". Hayashi-Samuel spaces are characterised, the value of the joint operations of $*$ and Ψ on the continuous function, completely continuous function, etc., is obtained, and the change of various kinds of generalised open sets under these operators is also found. It also studies the properties of these two set operators in the normed linear spaces.

The second chapter "PNDP-Manifolds" presence a new type of manifold called the partially negative-dimensional product manifold (PNDP-manifold), which is a special type of Einstein sequentially warped product manifold where the base manifold B is a Remannian (or pseudo-Riemannian) product manifold and the fibre manifold F is a derived smooth manifold.

In the third chapter "A Fibonacci Sequence Derived from Cardinalities of the Set of Open Sets of Khalimsky Topological Subspaces", Urysohn's type lemma for regular spaces is discussed using m-continuous mappings, and an overview of several versions of normal spaces is given.

In the fourth chapter "A Type of Compactness in Neutrosophic Topological Space via Grills", a Fibonacci sequence is derived from the cardinalities of the set of open sets of certain Khalimsky topological subspaces, along with the presence of certain features of the Khalimsky topological line.

In the fifth chapter "On $e^*\theta$-Regular and $e^*\theta$-Normal Spaces", examine compactness and weak G-compactness in a neutrosophic topological space through grills.

Also explore its fundamental characteristics and characterisation theorems and determine how it relates to other known neutrosophic compactness.

In sixth chapter "On e^*-Normal Spaces", a new class of regular spaces termed $e^*\theta$-regular spaces—a generalisation of the class of $\beta\theta$-regular spaces—is introduced. Several characterizations and fundamental features of both $e^*\theta$-regular and $e^*\theta$-normal spaces are examined.

In the seventh chapter "On Generalizations of H-Space in Ideal Topological Spaces", the class of e^*-normal spaces is studied using Ekici's concept of the e^*-open set. Additionally, the relationships between certain other types of normality that have been described in the literature, like s-normal space, p-normal space, and β-normal space, are examined.

A class of independent H-spaces that contains the class of all Hausdorff spaces is discussed in the eighth chapter "A Novel Mappings in Supra Topological Space with Neutrosophic Environment", along with the introduction and investigation of a new class of spaces that contains the class of all H-spaces.

The broad theory of ordinary relators is demonstrated in the ninth chapter "Invariant Study of Nets and Generalized Limits on Ideal Topological Spaces", which shows that superrelators cannot be used to generalise ordinary relators.

In the tenth chapter "On Associated Functions and Filters", two neutrosophic supra topological spaces are compared. The concept of a neutrosophic $\alpha\psi$-supra open set is incorporated into the neutrosophic continuous map and other maps. New functions are defined in the neutrosophic supra topological space, and their properties are examined.

The eleventh chapter "Hunting Cooperation and Fear Effect in Parallelism with Immature's Hunting in a Predator-Prey Symbiosis" presents the idea of limit points, which are essential to the study of many mathematical topics, including topology, functional analysis, metric space, and so on. It also examines how limit points and the points they are related to are preserved.

The twelfth chapter "Bianchi Type I Cosmological Model with Time-Dependent Cosmological Constant in Brans-Dicke Theory of Gravity" develops the true relational modification of a false but interesting lemma of Paweł Pasteczka on the constancy of the composition of certain set-valued functions.

In the thirteenth chapter "Bianchi Type-I Cosmological Model with a Special Law of Hubble Parameter in $f(R, T)$ Theory of Gravity", a few associated function results are examined, and their relationship to semigroup and hypergraph structures is established.

A thorough analysis of the relationship between a predator and its prey is conducted in the fourteenth chapter "Mathematical Modeling and Analysis of COVID-19: A New Trajectory in the Model Framework", after which the ability to hunt is granted to both the immature and mature stages of the predator. The impact of fear and the collaboration of mature predators during hunting are also covered. This chapter envisions a scenario in which the roles of the immature predator and the prey are reversed. As a result, the model is artificially constructed to exhibit three different kinds of biologically feasible equilibrium points, each of which has its stability analyses carried out; limit cycles are verified by means of theoretical

and graphical techniques; and the model's rich dynamical scenarios are visualised by means of numerical simulations, which demonstrate the significance of a few key parameters.

In the fifteenth chapter "Numerical Modelling in Heat Transfer Analysis of SWCNT/EG Nanofluid in a Stretching Sheet: A Darcy-Forchheimer Model", the evolution of pertinent cosmological parameters is studied, and the validity of the four energy conditions is verified. The study examines cosmic dynamics in a spatially homogeneous and anisotropic Bianchi type-I universe within the framework of the Brans-Dicke theory of gravity with a time-varying cosmological constant. Furthermore, cosmic dynamics is shown by the model, which is consistent with the available observational data.

The sixteenth chapter "Big Rip in Tsallis Holographic Dark Energy Model Within the Framework of General Relativity" study a spatially homogeneous and anisotropic Bianchi type-I universe filled with perfect fluid within the framework of $f(R,T)$ theory of gravity for the functional form $f(R,T) = f_1(R) + f_2(T)$, where λ is a constant, and obtains the exact solution of the gravitational field equations by using a special law for the Hubble parameter H. The evolution of relevant cosmological parameters and the physical behaviour of the model are investigated.

Using the functional form $f(R,T) = f_1(R) + f_2(T)$, where λ is a constant, the sixteenth chapter "Big Rip in Tsallis Holographic Dark Energy Model Within the Framework of General Relativity" studies a spatially homogeneous and anisotropic Bianchi type-I universe filled with perfect fluid within the framework of $f(R,T)$ theory of gravity. The exact solution of the gravitational field equations is obtained by using a special law for the Hubble parameter H. Examined are the changes in pertinent cosmological parameters and the model's physical behaviour.

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Closure Functions Induced by $*$ and Ψ Operators

Shyamapada Modak and Md. Monirul Islam

Abstract Through this chapter, we shall discuss the study of the joint operations of $*$ and Ψ of the ideal topological space $(X, \tau, \mathscr{I})$ with the help of the operators Ψ^* and $*^{\Psi}$, where

$$\Psi^*(A) = (\Psi(A))^* = \{x \in X : U_x \cap \Psi(A) \notin \mathscr{I}\}; \text{ and}$$
$$*^{\Psi}(A) = \Psi(A^*) = \Psi(\{x \in X : U_x \cap A \notin \mathscr{I}\}), \text{ where } U_x \in \tau(x), \text{ for } A \subseteq X.$$

From the operator Ψ^*, if $x \in \Psi^*(A)$, then no open set containing x belongs to $\mathscr{I}$. Using this fact, we characterize the Hayashi-Samuel spaces and obtain the value of joint operations of $*$ (resp. Ψ) and Ψ (resp. $*$) on the continuous function, completely continuous function etc. Further, we determine the change of various types of generalized open sets under these operators. In another section, we also discuss the properties of the above two set operators in the Normed Linear Spaces which is a part of Functional Analysis.

1 Introduction

The study of local function in ideal topological space $(X, \tau, \mathscr{I})$, where (X, τ) is a topological space and $\mathscr{I}$ is an ideal [14, 26] on X was introduced by Kuratowski [14] and Vaidyanathswamy [26]. The local function of a set $A \subseteq X$ with respect to a topology τ and an ideal $\mathscr{I}$ is denoted by A^*. If $\mathscr{I}$ contains $\emptyset$ only, then, A^* is nothing but the concept of limit points of A in (X, τ). Thus the local function is the generalization of the limit points. For $A \subseteq X$ in the ideal topological space $(X, \tau, \mathscr{I})$, A^* is closed in (X, τ), but $A^* \neq Cl(A)$, where $Cl(A)$ denotes the closure of A in (X, τ). A new topology has been obtained from $(X, \tau, \mathscr{I})$ with the help of

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local function and this topology is called $*$-topology [10] and it is finer than τ. The authors Hashimoto, Hayashi, Njåstad had obtained $*$-topology in different time but Janković and Hamlett [9, 13] and Modak and Bandyopadhyay [4, 18] have used modern notation of the $*$-topology and the notation is $\tau^*(\mathscr{I})$. In the topological space (X, τ), it is well known that $Int(A) = X\backslash Cl(X\backslash A)$, for $A \subseteq X$, where $Int(A)$ denotes the interior of A in (X, τ), in view of this, the author Natkaniec [21] has introduced another set operator Ψ and it is defined as: $\Psi(A) = X\backslash(X\backslash A)^*$, for a subset A of an ideal topological space $(X, \tau, \mathscr{I})$. Note that $\Psi(A)$ is an open set in (X, τ) but $\Psi(A) \neq Int(A)$. Note that the joint operations of $*$ (resp. Ψ) and Ψ (resp. $*$) on a subset A in the ideal topological space $(X, \tau, \mathscr{I})$ is not equal to the joint operations of Cl (resp. Int) and Int (resp. Cl) on A. It was established [12, 19] by the authors Modak and Islam.

Further, in this chapter we consider the joint operations of $*$ (resp. Ψ) and Ψ (resp. $*$) and we discussed various properties of the set valued set functions Ψ^* [19] and $*^{\Psi}$ [12]. We study these two set valued set functions in detail and obtain various results with the help of these two set functions. We also characterize Hayashi-Samuel space [6] with the help of Ψ^* and $*^{\Psi}$ set functions. The value of Ψ^* (resp. $*^{\Psi}$) on the open map, continuous map, strongly continuous map, completely continuous map, almost open map are also the part of this chapter. Further, we also give a brief discussion on dense sets and nowhere dense sets in ideal topological spaces with the help of these operators. Effects of these operators on normed linear (resp. metric) space are also a part of this chapter.

2 Preliminary

Followings are the necessary tools of this chapter.

Definition 1 ([14]) Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A be a subset of X. A set valued set function $()^* : \wp(X) \to \wp(X)$, defined as:

$$A^*(\mathscr{I}, \tau)(\text{or simply } A^*) = \{x \in X : U \cap A \notin \mathscr{I},\ U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau : x \in U\}$, is called local function of A with respect to τ and $\mathscr{I}$.

We will simply denote A^* for $A^*(\mathscr{I}, \tau)$.

Definition 2 ([21]) Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. The operator $\Psi : \wp(X) \to \wp(X)$ on A with respect to $\mathscr{I}$ and τ is

$$\Psi(A) = \{x \in X : \text{there exists } U \in \tau(x),\ U\backslash A \in \mathscr{I}\}.$$

It is also expressed by a nice relation $\Psi(A) = X\backslash(X\backslash A)^*$, for any $A \subseteq X$.

Definition 3 ([14]) A subset A of a topological space (X, τ) is said to be dense (resp. nowhere dense) if $Cl(A) = X$ (resp. $Int(Cl(A)) = \emptyset$).

Definition 4 ([8, 23]) Let X be a set, $\wp(X)$ be the power set of X and $cl : \wp(X) \to \wp(X)$ be any arbitrary set-valued set function. The function $cl(A)$, $A \subseteq X$, is called the closure of A and the pair (X, cl) is called a generalized closure space.

The closure function in a generalized closure space (X, cl) is called:

(a) grounded, if $cl(\emptyset) = \emptyset$,
(b) isotonic, if $A \subseteq B$ implies $cl(A) \subseteq cl(B)$,
(c) expanding, if $A \subseteq cl(A)$ for all $A \subseteq X$,
(d) sub-additive, if $cl(A \cup B) \subseteq cl(A) \cup cl(B)$,
(e) idempotent, if $cl(cl(A)) = cl(A)$,
(f) additive, if $\bigcup_{\lambda \in \Lambda} cl(A_\lambda) = cl(\bigcup_{\lambda \in \Lambda}(A_\lambda))$.

Definition 5 ([19]) The operator (or, set valued set function) $\Psi^* : \wp(X) \to \wp(X)$ on an ideal topological space $(X, \tau, \mathscr{I})$ is defined as:

$$\Psi^*(A) = (\Psi(A))^* = \{x \in X : U_x \cap \Psi(A) \notin \mathscr{I}\}, \text{ where } U_x \in \tau(x), \text{ for } A \subseteq X.$$

It is obvious that $(\Psi(A))^*$ is a closed set in $(X, \tau, \mathscr{I})$ for any $A \subseteq X$.

Definition 6 ([12]) The operator $*^\Psi : \wp(X) \to \wp(X)$ on an ideal topological space $(X, \tau, \mathscr{I})$ is defined as:

$$*^\Psi(A) = \Psi(A^*) = \Psi(\{x \in X : U_x \cap A \notin \mathscr{I}\}), \text{ for } A \subseteq X.$$

It is also represented as $*^\Psi(A) = X \backslash (X \backslash A^*)^*$, for $A \subseteq X$.

It is obvious that $*^\Psi(A)$ is an open set in $(X, \tau, \mathscr{I})$, for any $A \subseteq X$.

Definition 7 Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$, A is said to be regular open [24] (resp. Ψ^*-set [17], Ψ-C set [18], τ^*-dense [7], $*$-perfect [11], $\mathscr{I}$-dense [7]) if $A = Int(Cl(A))$ (resp. $A \subseteq (\Psi(A))^*$, $A \subseteq Cl(\Psi(A))$, $Cl^*(A) = X$, $A = A^*$, $A^* = X$).

Here $Cl^*(A)$ denotes the closure of A in the topological space $(X, \tau^*(\mathscr{I}))$.

Definition 8 ([20]) Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be an open map if for any open subset A of X, $f(A)$ is open in Y.

Definition 9 ([20]) Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be continuous if the inverse image of every open subset of Y is a open subset of X.

Definition 10 ([15]) Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be strongly continuous if for every subset A of X, $f(Cl(A)) \subseteq f(A)$.

Definition 11 ([3]) Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be completely continuous if the inverse image of every open subset of Y is a regular open subset of X.

Definition 12 ([2]) Let (X, τ) and (Y, σ) be topological spaces. A mapping f from X onto Y is called almost open if for each $y \in Y$, there exists an $x \in f^{-1}(y)$ such that there is an open base $\mathscr{B}$ at x such that $f(B)$ is open for each $B \in \mathscr{B}$.

Before considering the main results, we shall tell few words about the Hayashi-Samuel space. The condition $\tau \cap \mathscr{I} = \{\emptyset\}$ in the ideal topological space $(X, \tau, \mathscr{I})$ is called codense ideal according to the authors Dontchev et al. [7]. Newcomb [22]. Hamlett and Janković [9] call such ideal as τ-boundary whereas Dontchev [6] calls such spaces as "Hayashi-Samuel" spaces.

3 Ψ^* Operator

In this section, we shall study the properties of Ψ^* operator on ideal topological spaces as well as in Hayashi-Samuel spaces.

This operator has been introduced by Modak and Islam [19]. Here, we shall discuss the further properties of the operator Ψ^*.

Theorem 1 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, the operator $\Psi^* : \wp(X) \to \wp(X)$ is grounded.

Proof Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, $X = X^*$ [13]. Therefore, $\Psi^*(\emptyset) = (\Psi(\emptyset))^* = (X \backslash X^*)^* = (X \backslash X)^* = \emptyset^* = \emptyset$. Therefore, Ψ^* is grounded. □

The following example shows that the converse of the above theorem is not true.

Example 1 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{c\}, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Then, $(X, \tau, \mathscr{I})$ is not a Hayashi-Samuel space. We see that $(\Psi(\emptyset))^* = (X \backslash X^*)^* = (X \backslash \{a, b\})^* = (\{c\})^* = \emptyset$.

Theorem 2 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, the operator $\Psi^* : \wp(X) \to \wp(X)$ is isotonic.

Proof We know that the operators $()^*$ and Ψ are isotonic (or increasing). So, for any $A, B \subseteq X$ with $A \subseteq B$, we have $\Psi(A) \subseteq \Psi(B)$ and so $(\Psi(A))^* \subseteq (\Psi(B))^*$, that is, $\Psi^*(A) \subseteq \Psi^*(B)$. □

The operator Ψ^* is not expanding and it follows from the following example.

Example 2 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Let $A = \{c\}$. Then, $\Psi(A) = X \backslash (\{a, b\})^* = X \backslash X = \emptyset$. Therefore, $(\Psi(A))^* = \emptyset^* = \emptyset$. Thus, $\Psi^*(A) = \emptyset$ and $A \nsubseteq \Psi^*(A)$. Hence, Ψ^* is not expanding.

Following example shows that the operator Ψ^* is not subadditive.

Example 3 Consider $\mathbb{R}$ with usual topology and $\mathscr{I} = \{\emptyset\}$. Let $A = \mathbb{Q}$ and $B = \mathbb{R}\backslash\mathbb{Q}$. Then, $A \cup B = \mathbb{R}$. Also, $\Psi(A) = \Psi(\mathbb{Q}) = \mathbb{R}\backslash(\mathbb{R}\backslash\mathbb{Q})^* = \mathbb{R}\backslash\mathbb{R} = \emptyset$ and $\Psi(B) = \Psi(\mathbb{R}\backslash\mathbb{Q}) = \mathbb{R}\backslash\mathbb{Q}^* = \mathbb{R}\backslash\mathbb{R} = \emptyset$. So, $(\Psi(A))^* = (\Psi(B))^* = \emptyset$. Again, $(\Psi(A \cup B))^* = (\Psi(\mathbb{R}))^* = \mathbb{R}^* = \mathbb{R}$. Thus, $(\Psi(A \cup B))^* \nsubseteq (\Psi(A))^* \cup (\Psi(B))^*$, that is, $\Psi^*(A \cup B) \nsubseteq \Psi^*(A) \cup \Psi^*(B)$. Hence, Ψ^* is not subadditive.

However the following holds.

Theorem 3 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, for $A, B \in \wp(X)$, $\Psi^*(A) \cup \Psi^*(B) \subseteq \Psi^*(A \cup B)$.

Proof Let $A, B \in \wp(X)$. Since, the operator Ψ^* is isotonic and $A \subseteq A \cup B$, it follows that $\Psi^*(A) \subseteq \Psi^*(A \cup B)$. Similarly, $\Psi^*(B) \subseteq \Psi^*(A \cup B)$. These together imply that, $\Psi^*(A) \cup \Psi^*(B) \subseteq \Psi^*(A \cup B)$. □

From the Example 3, it follows that Ψ^* is not additive also.

Thus, we conclude that Ψ^*-operator is not a Kuratowski closure operator. Thus, it does not give a topology on X.

We know, if $Cl(Int(A)) = \emptyset$, for a subset A of a topological space (X, τ), then, $X\backslash Cl(Int(A)) = X$ and hence, $Int(Cl(X\backslash A)) = X$. Thus $X\backslash A$ is dense in X. Therefore, we have, $Cl(Int(A)) = \emptyset$, if and only if, $X\backslash A$ is dense in X. However, in an ideal topological space $(X, \tau, \mathscr{I})$, the following holds.

$\Psi^*(A) = \emptyset$ implies $(\Psi(A))^* = \emptyset$. Then, $(X\backslash(X\backslash A)^*)^* = \emptyset$ and $X^*\backslash(X\backslash A)^{**} = \emptyset$. This implies that $X^*\backslash(X\backslash A)^* = \emptyset$ [13] and hence, $(X\backslash A)^* \subseteq X^*$. Again if the space $(X, \tau, \mathscr{I})$ is a Hayashi-Samuel, the $X = X^*$ and we have $(X\backslash A)^* = X$. Thus for $\Psi^*(A) = \emptyset$, $X\backslash A$ is a $\mathscr{I}$-dense set and $Cl^*(X\backslash A) = (X\backslash A) \cup (X\backslash A)^* = X$. Thus, for Hayashi-Samuel space $(X, \tau, \mathscr{I})$, $\Psi^*(A) = \emptyset$ implies $X\backslash A$ is $*$-dense and $\mathscr{I}$-dense also.

Conversely, if $X\backslash A$ is dense in $(X, \tau^*(\mathscr{I}))$, then, $Cl^*(X\backslash A) = X$. Then, $X\backslash Int^*(A) = X$ and $A \cap \Psi(A) = \emptyset$ [9]. Thus, $\Psi(A) \subseteq X\backslash A$. Thus for the operator Ψ^*, if $X\backslash A$ is dense in $(X, \tau^*(\mathscr{I}))$, then, $\Psi^*(A) = \emptyset$ is not true always. But for $\mathscr{I}$-dense set $X\backslash A$, $\Psi^*(A) = \emptyset$. Therefore, we have, $\Psi^*(A) = \emptyset$ if and only if, $X\backslash A$ is $\mathscr{I}$-dense.

Following theorem shows that the value of $\Psi^*(A)$ is contained in A^*, for any $A \subseteq X$.

Theorem 4 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $A \subseteq X$, $\Psi^*(A) \subseteq A^*$.

Proof Let $A \subseteq X$. Since $(X, \tau, \mathscr{I})$ is a Hayashi-Samuel space, $\Psi(A) \subseteq A^*$ ([18]). This implies $(\Psi(A))^* \subseteq A^{**} \subseteq A^*$. □

Converse of the above theorem need not hold, in general.

Example 4 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathscr{I} = \{\emptyset, \{a\}\}$. Then, $(X, \tau, \mathscr{I})$ is not Hayashi-Samuel. Let $A = \{b\}$. Then, $\Psi^*(A) = \{b, c\} = A^*$.

Corollary 1 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $A \subseteq X$, $\Psi^*(A) \subseteq Cl^*(A)$.*

Proof Let $A \subseteq X$. Then by Theorem 4, $\Psi^*(A) \subseteq A^* \subseteq A \cup A^* = Cl^*(A)$. □

Corollary 2 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $A \subseteq X$, $\Psi^*(A) \subseteq Cl(A)$.*

Proof Let $A \subseteq X$. Then by Theorem 4, $\Psi^*(A) \subseteq A^* \subseteq Cl(A)$. □

Remark 1 The value of $\Psi^*(A)$, for a subset A of an ideal topological space $(X, \tau, \mathscr{I}_n)$ ($\mathscr{I}_n$ denotes the collection of all nowhere dense subsets in (X, τ)), is $\Psi^*(A) = (\Psi(A))^* = Int(Cl(Int(Cl(Int(Cl(A)))))) = Int(Cl(A))$. It is followed by [13, 19].

Theorem 5 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $U \in \tau$, then, $U^* \subseteq \Psi^*(U)$.

Proof Let $U \in \tau$. Then, by Theorem 2.1 of [18], $U \subseteq \Psi(U)$. This implies, $U^* \subseteq (\Psi(U))^*$. Hence, $U^* \subseteq \Psi^*(U)$. □

Theorem 6 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. If $U \in \tau$, then, $U^* = \Psi^*(U)$.

Proof Follows from Theorems 4 and 5. □

Corollary 3 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $U \in \tau$, $U \subseteq \Psi^*(U)$.*

Proof Let $U \in \tau$. Since $(X, \tau, \mathscr{I})$ is a Hayashi-Samuel space, therefore $U \subseteq U^*$ and $U \subseteq \Psi(U)$. These imply $U \subseteq U^* \subseteq (\Psi(U))^*$. Hence $U \subseteq \Psi^*(U)$. □

Corollary 4 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $U \in \tau$, $Cl^*(U) = \Psi^*(U)$.*

Proof Let $U \in \tau$. Then by Theorem 5 and by Corollary 3, $Cl^*(U) = U \cup U^* \subseteq \Psi^*(U)$. Again by Corollary 1, $\Psi^*(U) \subseteq Cl^*(U)$. Hence $Cl^*(U) = \Psi^*(U)$. □

Theorem 7 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $U \in \tau$, $\Psi^*(U) \supseteq Cl(U)$.

Proof Given that $U \in \tau$, then, $(\Psi(U))^* = Cl(\Psi(U))$ [18]. Again, by Theorem 6 and by [13], $\Psi^*(U) = (\Psi(U))^* \subseteq Cl(\Psi(U))$. □

Corollary 5 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $U \in \tau$, $\Psi^*(U) = Cl(\Psi(U)) = Cl^*(U)$.*

Theorem 8 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If each open set in $(X, \tau, \mathscr{I})$ is $*$-perfect, then, $\Psi^*(A) = \Psi(A)$, for any $A \subseteq X$.

Proof Let $A \subseteq X$. Then $\Psi(A)$ is open and hence $\Psi(A)$ is $*$-perfect. Therefore $(\Psi(A))^* = \Psi(A)$. Hence $\Psi^*(A) = \Psi(A)$. □

Theorem 9 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $J \in \mathscr{I}$, $\Psi^*(J) = \emptyset$.

Proof Let $J \in \mathscr{I}$. Then, $(\Psi(J))^* = (X\backslash(X\backslash J)^*)^* = (X\backslash X^*)^* = \emptyset$ (Since $J \in \mathscr{I}$, then, $(X\backslash J)^* = X^*$ [13]). □

Following example shows that the converse is not true.

Example 5 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathscr{I} = \{\emptyset, \{a\}\}$. Let $J = \{a\}$. Then, $\Psi^*(J) = \emptyset$ but X is not a Hayashi-Samuel space.

Theorem 10 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and $A \subseteq X$. Then, for $J \in \tau$, $\Psi^*(A \cup J) = \Psi^*(A\backslash J) = \Psi^*(A)$.

Proof Since $J \in \mathscr{I}$, so $\Psi(A \cup J) = \Psi(A\backslash J) = \Psi(A)$ ([9]). This implies $(\Psi(A \cup J))^* = (\Psi(A\backslash J))^* = (\Psi(A))^*$. Hence $\Psi^*(A \cup J) = \Psi^*(A\backslash J) = \Psi^*(A)$. □

A collection $m_X \subseteq \wp(X)$ is called a minimal structure on X, if $\emptyset, X \in m_X$. For minimal structure see [1, 16].

Remark 2 The collection $\{A \subseteq X : \Psi^*(X\backslash A) = X\backslash A\}$ contains $\emptyset$ and X, when the space $(X, \tau, \mathscr{I})$ is Hayashi-Samuel. Thus, the collection forms a minimal structure on X.

Note that the collection $\{A \subseteq X : \Psi^*(X\backslash A) = X\backslash A\}$ does not form a topology, since, the operator Ψ^* is not expanding, subadditive and additive, although this collection form a minimal structure on X.

Following theorems give the characterization of Hayashi-Samuel space.

Theorem 11 An ideal topological space $(X, \tau, \mathscr{I})$ is Hayashi-Samuel if and only if $\Psi^*(X) = X$.

Proof Let $(X, \tau, \mathscr{I})$ is Hayashi-Samuel. Then, $X^* = X$ [13]. Therefore, $\Psi^*(X) = (\Psi(X))^* = (X\backslash(X\backslash X)^*)^* = X^* = X$.

Conversely, let $\Psi^*(X) = X$. Then, $(\Psi(X))^* = X$ implies, $(X\backslash(X\backslash X)^*)^* = X$ or, $X^* = X$. Therefore, by [13], $(X, \tau, \mathscr{I})$ is Hayashi-Samuel. □

Theorem 12 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, $(X, \tau, \mathscr{I})$ is a Hayashi-Samuel space if and only if $\Psi^*(A) = Cl(\Psi(A))$, for each $A \subseteq X$.

Corollary 6 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, the following properties are equivalent:*

1. $\Psi^*(X) = X$;
2. $(X, \tau, \mathscr{I})$ *is a Hayashi-Samuel space;*
3. $\Psi^*(A) = Cl(\Psi(A))$, *for each* $A \subseteq X$.

Proof (1) implies (2): Follows from Theorem 11.

(2) implies (3): Follows from Theorem 12.

(3) implies (1): Let $\Psi^*(A) = Cl(\Psi(A))$ holds for each $A \subseteq X$. Then $\Psi^*(X) = Cl(\Psi(X)) = Cl[X\backslash(X\backslash X)^*] = Cl(X\backslash\emptyset^*) = Cl(X\backslash\emptyset) = Cl(X) = X$. Thus $\Psi^*(X) = X$. □

In the next section, we shall give a bunch of characterizations of Hayashi-Samuel space.

Theorem 13 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and (Y, σ) be a topological space. If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma)$ is a strongly continuous map, then, $f(\Psi^*(A)) \subseteq f(A)$.

Proof Let $A \subseteq X$. Then by Theorem 4, $\Psi^*(A) \subseteq A^*$ and since $A^* \subseteq Cl(A)$, therefore, $\Psi^*(A) \subseteq Cl(A)$. Hence $f(\Psi^*(A)) \subseteq f(Cl(A)) \subseteq f(A)$ (since f is strongly continuous). □

4 $*^{\Psi}$ Operator

In this section, we shall discuss the properties of $*^{\Psi}$ operator in ideal topological spaces as well as in Hayashi-Samuel spaces.

This operator has been introduced by Islam and Modak [12].

Followings are the detail study of the operator $*^{\Psi}$.

Example 6 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{c\}, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Then, $\Psi^*(X) = \{a, b\}$ and $*^{\Psi}(X) = X$.

This example shows that the values of the operators Ψ^* and $*^{\Psi}$ are not equal.

This operator is not grounded and it is followed by the following example.

Example 7 Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathscr{I} = \{\emptyset, \{a\}\}$. Then, $\Psi(\emptyset^*) = \Psi(\emptyset) = X\backslash X^* = X\backslash\{b, c, d\} = \{a\} \neq \emptyset$. Hence, $*^{\Psi}$ is not grounded.

Theorem 14 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, the operator $*^{\Psi} : \wp(X) \to \wp(X)$ is grounded if and only if $(X, \tau, \mathscr{I})$ is a Hayashi-Samuel space.

Proof By Theorem 6.1 of [13], $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space if and only if $X = X^*$.

Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, $*^{\Psi}(\emptyset) = \Psi(\emptyset^*) = \Psi(\emptyset) = X\backslash X^* = X\backslash X = \emptyset$. Hence, $*^{\Psi}$ is grounded.

Conversely, suppose $*^{\Psi}$ is grounded. Then, $*^{\Psi}(\emptyset) = \emptyset$. This implies $\Psi(\emptyset^*) = \emptyset$, and $X\backslash X^* = \emptyset$. Then, $X \subseteq X^*$. So $X = X^*$ and hence, $\tau \cap \mathscr{I} = \{\emptyset\}$. Thus, $(X, \tau, \mathscr{I})$ is a Hayashi-Samuel space. □

Further characterizations of Hayashi-Samuel spaces are as follows.

Corollary 7 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then the following properties are equivalent:*

1. *$(X, \tau, \mathscr{I})$ is a Hayashi-Samuel space;*
2. *$\Psi(\emptyset) = \emptyset$;*
3. *If $A \subseteq X$ is closed, then, $\Psi(A) \backslash A = \emptyset$;*
4. *$*^{\Psi} : \wp(X) \to \wp(X)$ is grounded;*
5. *if $A \subseteq X$, then, $Int(Cl(A)) = \Psi(Int(Cl(A)))$;*
6. *A is regular open, $A = \Psi(A)$;*
7. *if $U \in \tau$, then, $\Psi(U) \subseteq Int(Cl(U)) \subseteq U^*$;*
8. *if $I \in \mathscr{I}$, then, $\Psi(I) = \emptyset$;*
9. *$\Psi^*(X, \tau) = \Psi(X, \tau)$;*
10. *$\Psi^*(A) = Cl(\Psi(A))$, for each $A \subseteq X$;*
11. *$G \subseteq G^*$, for each $G \in \tau$;*
12. *$\Psi^*(X) = X$;*
13. *if $J \in \mathscr{I}$, then, $Int(J) = \emptyset$.*

Theorem 15 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, $*^{\Psi}(X) = X$.

Proof Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, $X^* = X$. So, $*^{\Psi}(X) = \Psi(X^*) = \Psi(X) = X$. □

Following example shows that the converse of the above theorem is not true.

Example 8 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{c\}, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Then, $*^{\Psi}(X) = \Psi(X^*) = \Psi(\{a, b\}) = X \backslash \{c\}^* = X$ but $(X, \tau, \mathscr{I})$ is not a Hayashi-Samuel space.

Theorem 16 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, the operator $*^{\Psi} : \wp(X) \to \wp(X)$ is isotonic.

Proof Let $A, B \subseteq X$ and $A \subseteq B$. Since both the operators $*$ and Ψ are isotonic, therefore $A^* \subseteq B^*$. This implies $\Psi(A^*) \subseteq \Psi(B^*)$. Thus $*^{\Psi}(A) \subseteq *^{\Psi}(B)$. Hence the operator $*^{\Psi}$ is isotonic. □

The operator $*^{\Psi}$ is not expanding and it follows from the following example.

Example 9 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Let $A = \{c\}$. Then, $A^* = \{c\}^* = \emptyset$. Therefore, $*^{\Psi}(A) = \Psi(A^*) = \Psi(\emptyset) = X \backslash X^* = X \backslash X = \emptyset$. Thus, $A \nsubseteq *^{\Psi}(A)$. Hence, $*^{\Psi}$ is not expanding.

The operator $*^{\Psi}$ is not subadditive and it follows from the following example.

Example 10 Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Let $A = \{a, c\}$ and $B = \{b, c\}$. Then, $A^* = \{a, c, d\}$, $B^* = \{b, c, d\}$ and $(A \cup B)^* = X$. Therefore, $\Psi(A^*) = \{a\}$, $\Psi(B^*) = \{b\}$ and $\Psi((A \cup B)^*) = X$. Thus, $*^{\Psi}(A \cup B) \nsubseteq *^{\Psi}(A) \cup *^{\Psi}(B)$. Hence, $*^{\Psi}$ is not subadditive.

However following holds

Theorem 17 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, for $A, B \in \wp(X)$, $*^{\Psi}(A) \cup *^{\Psi}(B) \subseteq *^{\Psi}(A \cup B)$.

Proof Let $A, B \in \wp(X)$. Since, the operator $*^{\Psi}$ is isotonic and $A \subseteq A \cup B$, it follows that $*^{\Psi}(A) \subseteq *^{\Psi}(A \cup B)$. Similarly, $*^{\Psi}(B) \subseteq *^{\Psi}(A \cup B)$. These together imply that, $*^{\Psi}(A) \cup *^{\Psi}(B) \subseteq *^{\Psi}(A \cup B)$. □

From the Example 10, it follows that $*^{\Psi}$ is not additive also.

Note that the operator $*^{\Psi}$ is not a Kuratowski closure operator.

Theorem 18 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $A \subseteq X$, $*^{\Psi}(A) \subseteq A^*$.

Proof Since A^* is a closed set, so by Corollary 7 (3), $\Psi(A^*) \backslash A^* = \emptyset$ implies $\Psi(A^*) \subseteq A^*$. Hence $*^{\Psi}(A) \subseteq A^*$. □

Corollary 8 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $A \subseteq X$, $*^{\Psi}(A) \subseteq Cl^*(A)$.*

Proof From Theorem 18, for $A \subseteq X$, $*^{\Psi}(A) \subseteq A^* \subseteq A \cup A^* = Cl^*(A)$. □

Theorem 19 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $*^{\Psi}(A) \subseteq A^*$, for $A \subseteq X$, then, $Cl^*(X \backslash A^*) \subseteq (X \backslash A^*)^*$.

Proof Given that $*^{\Psi}(A) \subseteq A^*$. Then, $\Psi(A^*) \subseteq A^*$. This implies that $X \backslash (X \backslash A^*)^* \subseteq A^*$. Thus, $X \backslash A^* \subseteq (X \backslash A^*)^*$. Therefore, $Cl^*(X \backslash A^*) \subseteq (X \backslash A^*)^*$. □

Remark 3 The value of $*^{\Psi}(A)$ for a subset A of an ideal topological space $(X, \tau, \mathscr{I}_n)$ is: $*^{\Psi}(A) = \Psi(A^*) = Int(Cl(Int(Cl(Int(Cl(A)))))) = Int(Cl(A))$.

Remark 4 For the space $(X, \tau, \mathscr{I}_n)$, $*^{\Psi}(A) = \Psi^*(X \backslash A)$, for any $A \subseteq X$.

Two operators $*^{\Psi}$ and Ψ^* have been related by the above when $\mathscr{I} = \mathscr{I}_n$. But they are not equal even $\mathscr{I} = \{\emptyset\}$. For the space $(X, \tau, \{\emptyset\})$, $\Psi^*(A) = Cl(\Psi(A))$ and $*^{\Psi}(A) = \Psi(Cl(A))$. More general version of the Remark 3 is given below.

Theorem 20 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $U \in \tau$, $*^{\Psi}(U) = Int(Cl(U))$.

Proof Given that $U \in \tau$, then, $\Psi(U^*) = \Psi(Cl(U))$. This implies that $\Psi(U^*) = X \backslash (X \backslash Cl(U))^* = X \backslash Cl(X \backslash Cl(U))$, since, $X \backslash Cl(U)$ is an open set. Then, $\Psi(U^*) = Int(Cl(U))$. □

Theorem 21 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then, for $U \in \tau$, $*^{\Psi}(U)$ is a regular open set.

Proof Let $U \in \tau$. Then, from Theorem 20, $*^{\Psi}(U) = IntCl(U)$. Then, $Int(Cl(*^{\Psi}(U))) = Int(Cl(Int(Cl(U)))) = Int(Cl(U)) = *^{\Psi}(U)$. Thus $*^{\Psi}(U)$ is a regular open set. □

Corollary 9 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and $A \subseteq X$ is regular open. Then, $*^{\Psi}(A) = \Psi(A)$.*

Proof Let $A \subseteq X$ be regular open. Then $A = Int(Cl(A))$ and A is open also. By Corollary 7 (6), $A = \Psi(A)$. By Theorem 20, $*^{\Psi}(A) = Int(Cl(A)) = A = \Psi(A)$. □

Corollary 10 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and $A \subseteq X$ is regular open. Then, $Int^*(A) = *^{\Psi}(A) \cap A$.*

Proof Since, A is regular open, therefore by Corollary 9, $*^{\Psi}(A) = \Psi(A)$. Hence $Int^*(A) = \Psi(A) \cap A = *^{\Psi}(A) \cap A$. □

Theorem 22 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and A be a $*$-perfect subset of X. Then $*^{\Psi}(A) = \Psi(A)$.

Proof Since A is $*$-perfect, therefore $A^* = A$. Then $*^{\Psi}(A) = \Psi(A^*) = \Psi(A)$. □

Remark 5 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Let each open subset of $(X, \tau, \mathscr{I})$ be $*$-perfect. Then for each $*$-perfect subset A of X, $*^{\Psi}(A) = \Psi^*(A)$.

Followings are the more sets for which the values of the operators $*^{\Psi}$ and Ψ^* are equal.

Theorem 23 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then for $J \in \mathscr{I}, *^{\Psi}(J) = X \backslash X^*$.

Proof Since, $J \in \mathscr{I}$, $J^* = \emptyset$ [13]. Thus $*^{\Psi}(J) = \Psi(J^*) = \Psi(\emptyset) = X \backslash X^*$. □

Corollary 11 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then for $J \in \mathscr{I}, *^{\Psi}(J) = \emptyset$.*

Remark 6 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space. Then the value of $*^{\Psi}(J)$ and $\Psi^*(J)$ are equal for each $J \in \mathscr{I}$.

Theorem 24 Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then for $J \in \mathscr{I}, *^{\Psi}(A \cup J) = *^{\Psi}(A \backslash J) = *^{\Psi}(A)$.

Proof Let $A \subseteq X$ and $J \in \mathscr{I}$. Then $(A \cup J)^* = (A \backslash J)^* = A^*$ [13]. Therefore $\Psi((A \cup J)^*) = \Psi((A \backslash J)^*) = \Psi(A^*)$. This implies $*^{\Psi}(A \cup J) = *^{\Psi}(A \backslash J) = *^{\Psi}(A)$. □

Following is the remarkable relation for the operators $*^{\Psi}$ and Ψ^*.

Theorem 25 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then for $A \subseteq X, *^{\Psi}(A) = X \backslash \Psi^*(X \backslash A)$.

Proof Let $A \subseteq X$. Then, $\Psi^*(X \backslash A) = (\Psi(X \backslash A))^* = [X \backslash (X \backslash (X \backslash A))^*]^* = (X \backslash A^*)^* = X \backslash [X \backslash (X \backslash A^*)^*] = X \backslash \Psi(A^*) = X \backslash *^{\Psi}(A)$. Therefore, $*^{\Psi}(A) = X \backslash \Psi^*(X \backslash A)$. □

The relation of the Theorem 25 is similar to Natkaniec's relation $\Psi(A) = X\backslash(X\backslash A)^*$. From [12, 19], we know that $Int(Cl(A)) \neq *^{\Psi}(A)$ and $Cl(Int(A)) \neq \Psi^*(A)$ but with together, they obey the same relation like, $Cl(Int(A)) = X\backslash Int(Cl(X\backslash A))$.

Corollary 12 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, for $A \subseteq X$, $\Psi^*(A) = X\backslash *^{\Psi}(X\backslash A)$.*

We know from literature that the collection $\{A \subseteq X : Int(Cl(A)) = \emptyset\}$ for a topological space (X, τ) forms an ideal on X. But it is a relevant question for us that the collection $\{A \subseteq X : *^{\Psi}(A) = \emptyset\}$ for an ideal topological space $(X, \tau, \mathscr{I})$ forms an ideal on X. Following is the partial answer of this question.

Example 11 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\mathscr{I} = \{\emptyset, \{a\}\}$. Then, $\Psi(\emptyset^*) = \Psi(\{a\}^*) = \Psi(\{c\}^*) = \Psi(\{a, c\}^*) = \{a\}$, $\Psi(\{b\}^*) = \Psi(\{a, b\}^*) = \Psi(\{b, c\}^*) = \Psi(X^*) = X$.

We have shown that, the collection $\{A \subseteq X : *^{\Psi}(A) = \emptyset\}$ is not necessarily an ideal on X. However, the following example shows that the same collection forms an ideal and this ideal is larger than the original ideal.

Example 12 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{b, c\}, X\}$, $\mathscr{I} = \{\emptyset, \{a\}\}$. Then, $\{A \subseteq X : *^{\Psi}(A) = \emptyset\} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$.

Now we look at the value of $*^{\Psi}$ on functions.

Corollary 13 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and (Y, σ) be a topological space. If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma)$ is a continuous function, then, $*^{\Psi}(f^{-1}(U))$ is a regular open in $(X, \tau, \mathscr{I})$ for each open set U in (Y, σ).*

Proof Let $U \in \sigma$. Since f is continuous, so $f^{-1}(U) \in \tau$. By Theorem 21, $*^{\Psi}(f^{-1}(U))$ is regular open in $(X, \tau, \mathscr{I})$. □

Corollary 14 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and (Y, σ) be a topological space. If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma)$ is a completely continuous function, then, $*^{\Psi}(f^{-1}(U))$ is a regular open in $(X, \tau, \mathscr{I})$ for each open set U in (Y, σ).*

Proof Let U be open in (Y, σ). Since f is completely continuous, therefore $f^{-1}(U)$ is regular open in (X, τ) and hence open in (X, τ). Then by Theorem 21, $*^{\Psi}(f^{-1}(U))$ is a regular open in $(X, \tau, \mathscr{I})$. □

Corollary 15 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and (Y, σ) be a topological space. If $f : (Y, \sigma) \to (X, \tau, \mathscr{I})$ is an open map, then, $*^{\Psi}(f(U))$ is a regular open set in $(X, \tau, \mathscr{I})$ for each open set U in (Y, σ).*

Proof Let U be an open set in Y. Since f is an open map, therefore $f(U)$ is open in X. Then by Theorem 21, $*^{\Psi}(f(U))$ is a regular open set in $(X, \tau, \mathscr{I})$. □

Corollary 16 *Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and (Y, σ) be a topological space. If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma)$ is almost open, then, $f(*^{\Psi}(A))$ is open in (Y, σ) for each $A \subseteq X$.*

Proof Follows from the definition of almost open mapping and the fact that $*^{\Psi}(A)$ is an open set, for any $A \subseteq X$. □

Theorem 26 Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel space and (Y, σ) be a topological space. If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma)$ is a strongly continuous map, then, $f(*^{\Psi}(A)) \subseteq f(A)$.

Proof Let $A \subseteq X$. Then by Theorem 18, $*^{\Psi}(A) \subseteq A^*$ and since $A^* \subseteq Cl(A)$, therefore, $*^{\Psi}(A) \subseteq Cl(A)$. Hence $f(*^{\Psi}(A)) \subseteq f(Cl(A)) \subseteq f(A)$ (since f is strongly continuous). □

5 Ψ^* and $*^{\Psi}$ Operators on Normed Linear Spaces

In this section, we redefine the operators $()^*$, Ψ, Ψ^* and $*^{\Psi}$ on Normed Linear Space (NLS) (resp. Metric space). As every NLS is metrizable Space (MS), thus we can extend the operators on NLS. An algebraic structure (field) has been included in the NLS $(X, \|.\|)$ (or Hilbert Space). Thus for the study of ideal on normed liner space, there is a question that "where do we construct the ideal?" That is, it is a collection of subsets of the field or X. We are avoiding the chaotic situation by considering the ideal on the original space X. Then we shall study the above mentioned operators on NLS (or Hilbert Space).

In a metric space (X, d) or NLS $(X, \|.\|)$ and an ideal $\mathscr{I}$ on X, we consider following four operators through the open ball [25],

$U(x, r) = \{y \in X : \|x - y\| < r\}$, where $x \in X$ and $r > 0$.

1. $E^* = \{x \in X : U(x, r) \cap E \notin \mathscr{I}\}$, where $E \subseteq X$.
2. $\Psi(E) = \{x \in X :$ there exists a $U(x, r)$, $r > 0$ such that $U(x, r) \backslash E \in \mathscr{I}\}$, where $E \subseteq X$.
3. $\Psi^*(E) = \{x \in X : U(x, r) \cap \Psi(E) \notin \mathscr{I}$ for every $r > 0\}$, where $E \subseteq X$.
4. $*^{\Psi}(E) = X \backslash \Psi^*(X \backslash E)$, where $E \subseteq X$.

A metric space (X, d) or NLS $(X, \|.\|)$ and an ideal $\mathscr{I}$ on X is denoted as $(X, d, \mathscr{I})$ or $(X, \|.\|, \mathscr{I})$ and called it ideal metric space or ideal normed linear space.

Definition 13 An ideal NLS $(X, \|.\|, \mathscr{I})$ is said to be J-ideal space if it has the property:

$$P : \text{ "}E^* \neq \emptyset \text{ for some } E \in \wp(X)\text{"}.$$

Lemma 1 *For a J-ideal NLS $(X, \|.\|, \mathscr{I})$, $\mathscr{I}$ is a proper ideal.*

Converse of the lemma need not hold in general.

Example 13 Let $X = \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers. Define d on X by:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Consider $\mathscr{I} = \mathscr{I}_f$ (the collection of all finite subsets of $\mathbb{N}$). Then $\mathscr{I}_f$ is a proper ideal on $\mathbb{N}$. It is crystal clear that $\mathbb{N}^* = \emptyset$ and thats why $E^* = \emptyset$, for any $E \in \wp(\mathbb{N})$.

Theorem 27 Let $(X, \|.\|, \mathscr{I})$ be an ideal NLS and $E \subseteq X$. Then $x \in *^{\Psi}(E)$ if and only if there exists $U(x, r),\ r > 0$ such that $U(x, r) \backslash E^* \in \mathscr{I}$.

Proof Given that there exists $U(x, r),\ r > 0,\ U(x, r) \cap \Psi(X \backslash E) \in \mathscr{I}$. Thus $U(x, r) \backslash E^* \in \mathscr{I}$.

If we reverse the logic, then we get the converse part. □

Now we introduce filter from [5].

Let X be a nonempty set and $\mathscr{F} \subseteq \wp(X)$. Then $\mathscr{F}$ is called a filter on X if it satisfies the followings:

(i) $\emptyset \notin \mathscr{F}$,
(ii) $B \in \mathscr{F}$ and $B \subseteq A$ implies $A \in \mathscr{F}$,
(iii) $A,\ B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$.

Let $\mathscr{I}$ be a proper ideal on a set X. Then $\mathscr{F} = \{A \subseteq X :\ X \backslash A \in \mathscr{I}\}$ forms a filter on X. This filter is called the associated filter on X and denoted as $\mathscr{F}_{\mathscr{I}}$.

Corollary 17 *Let $(X, \|.\|, \mathscr{I})$ be a J-ideal NLS and $E \subseteq X$. If $x \in *^{\Psi}(E)$, then the associated filter $\mathscr{F}_{\mathscr{I}} \nrightarrow x$.*

Proof Given that there exists $U(x, r),\ r > 0, U(x, r) \backslash E^* \in \mathscr{I}$. Thus $X \backslash U(x, r) \cup E^* \in \mathscr{F}_{\mathscr{I}}$ and hence $X \backslash U(x, r) \in \mathscr{F}_{\mathscr{I}}$. Hence $\mathscr{F}_{\mathscr{I}} \nrightarrow x$. □

Corollary 18 *Let $(X, \|.\|, \mathscr{I})$ be a J-ideal NLS and $E \subseteq X$. If $x \in *^{\Psi}(E)$, then x is not a cluster point of the associated filter $\mathscr{F}_{\mathscr{I}}$.*

Theorem 28 Let $(X, \|.\|, \mathscr{I})$ be an ideal NLS and $E \subseteq X$. If $x \in *^{\Psi}(E)$, then there is no $U(x, r),\ r > 0$ such that $U(x, r) \in \mathscr{I}$.

Proof Given that $U(x, r) \cap \Psi(E) \notin \mathscr{I}$ for every $r > 0$. So $U(x, r) \notin \mathscr{I}$ for every $r > 0$. Hence the result. □

Corollary 19 *Let $(X, \|.\|, \mathscr{I})$ be a J-ideal NLS and $E \subseteq X$. If $x \in *^{\Psi}(E)$, then*

1. *the associated filter $\mathscr{F}_{\mathscr{I}} \nrightarrow x$;*
2. *x is not a cluster point of the associated filter $\mathscr{F}_{\mathscr{I}}$.*

One of the special type of ideal is given below.

An ideal $\mathscr{I}_u$ on a set X is called a universal ideal if, for any $A \subseteq X$, either $A \in \mathscr{I}_u$ or $X \backslash A \in \mathscr{I}_u$.

Corollary 20 *Let $(X, \|.\|, \mathscr{I}_u)$ be an ideal NLS, where $\mathscr{I}_u$ is a universal ideal on X. Let $E \subseteq X$ and $x \in *^{\Psi}(E)$. Then*

1. *the associated filter $\mathscr{U} \nrightarrow x$,*
2. *x is a cluster point of the associated filter $\mathscr{U}$.*

In respect of continuous linear functionals, the function related results of the earlier sections can be improved by the following way.

Theorem 29 Let $(X, \|.\|, \mathscr{I})$ be an ideal NLS and (Y, σ) be a topological space. Suppose $f \in X'$ (collection of continuous linear functionals) and $g : K \to Y$ is a continuous function, then $*^{\psi}((f \circ g)^{-1}(U))$ is a regular open set in $(X, \|.\|, \mathscr{I})$.

6 Conclusion

This chapter finds the conditions for equality of the role of the joint operations of the operators $*$ (resp. Ψ) and Ψ (resp. $*$) with the joint operators Closure (resp. Interior) and Interior (resp. Closure). Various types of closure spaces are obtained from Ψ^* and $*^{\Psi}$ operators. Role of these operators in Normed Linear Spaces have also been obtained here.

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PNDP-Manifolds

Alexander Pigazzini, Cenap Özel, and Saeid Jafari

Abstract We introduce a novel concept known as the partially negative dimensional product manifold, abbreviated as PNDP-manifold. Specifically, a PNDP-manifold represents a unique form of Einstein sequential warped product manifold, where the base manifold $\mathscr{B}$ is a Riemannian (or pseudo-Riemannian) product manifold expressed as $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$, with $\mathscr{B}_2$ being an Einstein manifold. The fiber manifold $\mathscr{F}$, on the other hand, is treated as a derived smooth manifold, characterized by a Kuranishi neighborhood $(\mathbb{R}^d, \mathscr{E}, S)$, where $\mathscr{E}$ serves as the obstruction bundle and S is a smooth section. This setup allows for the possibility of a "virtual" dimension, which may even be negative. From the perspective of differential geometry, this particular type of Einstein sequential warped product manifold offers a broader range of exact solutions to Einstein's field equations, while keeping the computations relatively straightforward, especially when compared to Einstein warped-product manifolds with Ricci-flat fibers $(\mathscr{F}; \ddot{g})$. On a more speculative level, by interpreting the fiber as a derived smooth manifold, the dimensions of a PNDP-manifold extend beyond the traditional geometric understanding of dimension, being instead viewed as "virtual" dimensions. With appropriate interpretation, this concept opens up the possibility of new "hidden" dimensions, which could have intriguing speculative and practical implications. These may include applications in fields like econophysics, where financial markets influenced by ghost fields such as dark volatility could be modeled, or in cosmology, where the idea of "emerging spaces" might be explored.

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1 Introduction

In this work, we explore a specific type of Einstein sequential warped product manifold. Here, the fiber manifold $\mathscr{F}$ is treated with the tools of derived geometry. To illustrate this, consider two transversal submanifolds $N_1 \to W$ and $N_2 \to W$ with codimensions c_1 and c_2, respectively. Their intersection Q naturally forms a submanifold with codimension $c_1 + c_2$. Derived geometry, however, allows us to move beyond the requirement of transversality, enabling us to interpret a non-transversal intersection Q as a derived smooth manifold with codimension $c_1 + c_2$. This means that the dimension of Q, given by $\dim(Q) - c_1 - c_2$, can be negative. Hence, the dimensions derived in this manner are not confined to the conventional geometric interpretation of "dimension" but can be understood as "virtual" dimensions.

In this paper, we will consider the fiber $\mathscr{F}$ as a derived smooth manifold, specifically within the context of a Kuranishi neighborhood $(\mathbb{R}^d, \mathscr{E}, S)$.

2 Preliminaries

Let's begin with a brief introduction to warped product manifolds, a concept introduced by Bishop and O'Neill [3] to construct various manifolds with negative curvature. For further insights into this topic, we recommend consulting [3, 4].

Let $(\mathscr{B}, g_{\mathscr{B}})$ and $(\mathscr{F}, g_{\mathscr{F}})$ be two Riemannian manifolds and $\pi_{\mathscr{B}}$, $\pi_{\mathscr{F}}$ be the projection of $\mathscr{B} \times \mathscr{F}$ onto $\mathscr{B}$ and $\mathscr{F}$, respectively.

Definition 1 The warped product $\mathscr{M} = \mathscr{B} \times_\xi \mathscr{F}$ is the manifold $\mathscr{B} \times \mathscr{F}$ provided with the metric tensor $g = \pi_{\mathscr{B}}^* g_{\mathscr{B}} + \xi^2 \pi_{\mathscr{F}}^* g_{\mathscr{F}}$, where * denotes the pullback and ξ is a positive smooth function on $\mathscr{B}$, the so-called warping function.

Let p and q be a points in $\mathscr{B}$ and $\mathscr{F}$ respectively. If Γ is tangent to $\mathscr{B} \times \mathscr{F}$, the we have:

$$\langle \Gamma, \Gamma \rangle = \langle d\pi_{\mathscr{B}}(\Gamma), d\pi_{\mathscr{B}}(\Gamma) \rangle + \xi^2(p)(d\pi_{\mathscr{F}}(\Gamma), d\pi_{\mathscr{F}}(\Gamma)).$$

The manifold $\mathscr{B}$ is referred to as the base-manifold of $\mathscr{M} = \mathscr{B} \times_\xi \mathscr{F}$, while $\mathscr{F}$ is known as the fiber-manifold. When the warping function $\mathscr{F}$ is equal to 1, the warped product $\mathscr{B} \times_\xi \mathscr{F}$ simplifies to the standard Riemannian product manifold. The submanifolds $\mathscr{B} \times \{q\} = \pi_{\mathscr{F}}^{-1}(q)$ and $\{p\} \times \mathscr{F} = \pi_{\mathscr{B}}^{-1}(p)$ within $\mathscr{M}$ are Riemannian submanifolds, termed leaves and fibers, respectively. Vectors that are tangent to the leaves are called horizontal vectors, and those tangent to the fibers are called vertical vectors. The operator $\mathscr{O}$ represents the orthogonal projection of $T_{(p,q)}\mathscr{M}$ onto the horizontal subspace $T_{(p,q)}(\mathscr{B} \times \{q\})$, while $\mathscr{P}$ denotes the projection onto the vertical subspace $T_{(p,q)}(\{p\} \times \mathscr{F})$.

Lemma 1 *If $h \in \mathscr{Q}(\mathscr{B})$, where $\mathscr{Q}(\mathscr{B})$ is the set of all smooth real-valued functions on $\mathscr{M}$, then the gradient of the lift $h \circ \pi_{\mathscr{B}}$ of h to $\mathscr{M} = \mathscr{B} \times_\xi \mathscr{F}$ is the lift to $\mathscr{M}$ of the gradient of h on $\mathscr{B}$.*

Proof If v is a vertical tangent vector to $\mathscr{M}$, then $\langle \nabla(h \circ \pi_{\mathscr{B}}), v\rangle = v(h \circ \pi_{\mathscr{B}}) = d\pi_{\mathscr{B}}(v)h = 0$, since $d\pi_{\mathscr{B}}(v) = 0$. Thus $\nabla(h \circ \pi_{\mathscr{B}})$ is horizontal. If w is horizontal,

$$\begin{aligned}\langle d\pi_{\mathscr{B}}(\nabla(h \circ \pi_{\mathscr{B}})), d\pi_{\mathscr{B}}(w)\rangle &= \langle \nabla(h \circ \pi_{\mathscr{B}}), w\rangle = w(h \circ \pi_{\mathscr{B}}) = d\pi_{\mathscr{B}}(w)h \\ &= \langle \nabla h, d\pi_{\mathscr{B}}(w)\rangle, \end{aligned} \tag{1}$$

which implies that at each point, $d\pi_{\mathscr{B}}(\nabla(h \circ \pi_{\mathscr{B}})) = \nabla h$. □

The following proposition is to define the Levi-Civita connection ∇ of $\mathscr{M} = \mathscr{B} \times_{\xi} \mathscr{F}$ related to those of $\mathscr{B}$ and $\mathscr{F}$.

Proposition 1 *Consider the manifold $\mathscr{M} = \mathscr{B} \times_{\xi} \mathscr{F}$. If $\Gamma, \Upsilon \in \mathscr{S}(\mathscr{B})$ and $G, V \in \mathscr{S}(\mathscr{F})$, where $\mathscr{S}(\mathscr{B})$ represents the set of all lifts on $\mathscr{B}$ to $\mathscr{B} \times \mathscr{F}$ and $\mathscr{S}(\mathscr{F})$ denotes the set of all lifts on $\mathscr{F}$ to $\mathscr{B} \times \mathscr{F}$, then the following properties hold:*

(a) The covariant derivative $\nabla_{\Gamma}\Upsilon \in \mathscr{S}(\mathscr{B})$ corresponds to the lift of $\nabla_{\Gamma}\Upsilon$ on $\mathscr{B}$.
(b) The covariant derivatives satisfy $\nabla_{\Gamma}U = \nabla_{U}\Gamma = (\Gamma \ln \xi)U$.
(c) The normal component of $\nabla_G V$, denoted as $nor(\nabla_G V)$, is given by $\eta(G, V) = -\frac{\langle U, V\rangle}{\xi}\nabla\xi$, where $\eta(G, V)$ represents the second fundamental form on the fiber.
(d) The tangential component $tan(\nabla_G V) \in \mathscr{S}(\mathscr{F})$ corresponds to the lift of $\ddot{\nabla}_G V$ on $\mathscr{F}$, where $\ddot{\nabla}$ is the Levi-Civita connection on $\mathscr{F}$.

Proof Starting with the Koszul formula for $2\langle \nabla_{\Gamma}\Upsilon, G\rangle$, we reduce it to $\langle G, [\Gamma, \Upsilon]\rangle - G\langle \Gamma, \Upsilon\rangle$ since $[\Gamma, G] = [\Upsilon, G] = 0$. Given that Γ and Υ are lifts from $\mathscr{B}$, $\langle \Gamma, \Upsilon\rangle$ remains constant on the fibers. Since G is vertical, $G\langle \Gamma, \Upsilon\rangle = 0$. Moreover, as $[\Gamma, \Upsilon]$ is tangent to the leaves, $\langle G, [\Gamma, \Upsilon]\rangle = 0$. Consequently, $\langle \nabla_{\Gamma}\Upsilon, G\rangle = 0$ for all $G \in \mathscr{S}(\mathscr{F})$. This implies that $\nabla_{\Gamma}\Upsilon$ is horizontal, and because $\pi_{\mathscr{B}}|_{\mathscr{B}\times q}$ is an isometry, property (a) follows.

Since $[\Gamma, G] = 0$, we conclude $\nabla_{\Gamma}G = \nabla_{G}\Gamma$. Given that these vector fields are vertical, $\langle \nabla_{\Gamma}G, Y\rangle = -\langle G, \nabla_{\Gamma}\Upsilon\rangle = 0$. The only term that survives in the Koszul formula for $2\langle \nabla_{\Gamma}G, V\rangle$ is $\Gamma\langle G, V\rangle$.

Thus:

$$(e) \quad 2\langle \nabla_{\Gamma}G, V\rangle = \Gamma\langle G, V\rangle. \tag{2}$$

Using the definition of the warped product metric, we have $\langle G, V\rangle_{(p,q)} = \xi^2(p)\langle G_q, V_q\rangle$. By substituting ξ with $\xi \circ \pi_{\mathscr{B}}$, we get $\langle G, V\rangle = \xi^2\langle (G, V) \circ \pi_{\mathscr{F}}\rangle$. Therefore, $\Gamma\langle U, V\rangle = X[\xi^2\langle (G, V) \circ \pi_{\mathscr{F}}\rangle] = 2\xi X[\xi\langle (G, V) \circ \pi_{\mathscr{F}}\rangle] = 2(\Gamma \ln \xi)\langle G, V\rangle$. Combining this with equation (e) establishes property (b). Which then leads to the following:

$$\langle \nabla_G V, \Gamma\rangle = -\langle V, \nabla_G\Gamma\rangle = -(\Gamma \ln \xi)\langle G, V\rangle. \tag{3}$$

By utilizing *Lemma* 1, we get $\Gamma\xi = \langle \nabla\xi, \Gamma\rangle$ on $\mathscr{M}$ as on $\mathscr{B}$. Thus, for any Γ, we obtain:

$$\langle \nabla_G V, \Gamma \rangle \xi = -\langle G, V \rangle \langle \nabla \xi, \Gamma \rangle, \tag{4}$$

which confirms property (c).

Since G and V are tangent to all fibers, $\tan(\nabla_G V)$ denotes the fiber covariant derivative used to the restrictions of G and V on that fiber. This proves property (d). □

The next result provides the curvature ($\bar{Riem}$) of a warped product $\mathscr{M} = \mathscr{B} \times_\xi \mathscr{F}$ in terms of its warping function ξ and the curvature tensors $Riem$ and $\ddot{Riem}$ of $\mathscr{B}$ and $\mathscr{F}$ respectively.

Proposition 2 *Let $\mathscr{M} = \mathscr{B} \times_\xi \mathscr{F}$ be a warped product of two (pseudo-)Riemannian manifolds. If $\Gamma, \Upsilon, Z \in \mathscr{S}(\mathscr{B})$ and $G, V, W \in \mathscr{S}(\mathscr{F})$, then the following properties hold:*

(f) $\bar{Riem}(\Gamma, \Upsilon)Z \in \mathscr{S}(\mathscr{B})$ *is the lift of* $Riem(\Gamma, \Upsilon)Z$ *on* $\mathscr{B}$.
(g) $\bar{Riem}(\Gamma, V)\Upsilon = \frac{H^\xi(\Gamma, \Upsilon)}{\xi} V$,
(h) $\bar{Riem}(\Gamma, \Upsilon)V = \bar{Riem}(V, W)\Gamma = 0$,
(i) $\bar{Riem}(\Gamma, V)W = -\frac{\langle V, W \rangle}{\xi} \nabla_\Gamma(\nabla \xi)$,
(j) $\bar{Riem}(V, W)G = \ddot{Riem}(V, W)G + \frac{\langle \nabla \xi, \nabla \xi \rangle}{\xi^2} (\langle V, G \rangle W - \langle W, G \rangle V)$,
where H^ξ denotes the Hessian of ξ.

Proof The projection $\pi_{\mathscr{B}} : \mathscr{M} \to \mathscr{B}$ is isometric on each leaf, which implies that the curvature tensor $Riem$ of each leaf corresponds to the curvature tensor $\bar{Riem}$ of $\mathscr{M}$ on horizontal vectors. Since the leaves are totally geodesic in $\mathscr{M}$, this gives us property (f).

For $\Gamma, \Upsilon \in \mathscr{S}(\mathscr{B})$ and $V \in \mathscr{S}(\mathscr{F})$, since $[\Gamma, \Upsilon] = 0$ and $Riem(\Gamma, V)\Upsilon = \nabla_\Gamma \nabla_V \Upsilon - \nabla_V \nabla_\Gamma \Upsilon$, by Proposition 1:

$$\begin{aligned} \nabla_\Gamma \nabla_V \Upsilon = \nabla_\Gamma(\Upsilon(\ln \xi)V) &= (\Gamma \Upsilon(\ln \xi))V + (\Upsilon(\ln \xi))\nabla_\Gamma V && (5) \\ &= \{\Gamma \Upsilon(\ln \xi) + (\Upsilon \xi) X \xi^{-1}\} V + (\Gamma(\ln \xi))(\Upsilon(\ln \xi))V && (6) \\ &= (\Gamma \Upsilon(\ln \xi))V. && (7) \end{aligned}$$

Thus,

$$(k) \quad \bar{Riem}(\Gamma, V)\Upsilon = (\Gamma \Upsilon(\ln \xi))V - \nabla_V \nabla_\Gamma \Upsilon. \tag{8}$$

However, since $\nabla_\Gamma \Upsilon \in \mathscr{S}(\mathscr{B})$, we have $\nabla_V \nabla_\Gamma \Upsilon = (\nabla_\Gamma \Upsilon(\ln \xi))V$, and combining this with (k) gives us property (g).

To prove (h), assume $[V, W] = 0$,

$$\nabla_V \nabla_W \Gamma = (V\Gamma(\ln\xi))W + (\Gamma(\ln\xi))\nabla_V W, \tag{9}$$

but $\Gamma(\ln\xi)$ is constant on the fibers, so $V\Gamma(\ln\xi) = 0$. Therefore,

$$\bar{Riem}(V, W)\Gamma = (\Gamma(\ln\xi))(\nabla_W V - \nabla_V W) = \Gamma(\ln\xi)[W, V] = 0. \tag{10}$$

By the symmetry of the curvature tensor, $\langle \bar{Riem}(\Gamma, \Upsilon)V, W\rangle = \langle \bar{Riem}(V, W)\Gamma, \Upsilon\rangle = 0$.

From (f), $\langle \bar{Riem}(\Gamma, \Upsilon)V, Z\rangle = -\langle \bar{Riem}(\Gamma, \Upsilon)Z, V\rangle = 0$. These equations hold for all $W \in \mathscr{S}(\mathscr{F})$ and $Z \in \mathscr{S}(\mathscr{B})$, hence $\bar{Riem}(\Gamma, \Upsilon)V = 0$.

To prove (i), we note that $\bar{Riem}(\Gamma, V)W$ is horizontal, as

$$\langle \bar{Riem}(\Gamma, V)W, G\rangle = \langle \bar{Riem}(W, G)\Gamma, V\rangle = 0, \tag{11}$$

according to (h). Since $\bar{Riem}(V, W)\Gamma = 0$, it follows from the first Bianchi identity that

$$\bar{Riem}(\Gamma, V)W = \bar{Riem}(\Gamma, W)V. \tag{12}$$

Using (g),

$$\langle \bar{Riem}(\Gamma, V)W, \Upsilon\rangle = \langle \bar{Riem}(V, \Gamma)\Upsilon, W\rangle = -\frac{H^\xi(\Gamma, \Upsilon)}{\xi}\langle V, W\rangle \tag{13}$$

$$= -\frac{\langle V, W\rangle}{\xi}\langle \nabla_\Gamma(\nabla\xi), \Upsilon\rangle. \tag{14}$$

Since $\bar{Riem}(\Gamma, V)W$ is horizontal and the equation holds for all Υ, we obtain property (i).

For property (j), observe that $\bar{Riem}(V, W)G$ is a vertical vector field because $\langle \bar{Riem}(V, W)G, \Gamma\rangle = -\langle \bar{Riem}(V, W)\Gamma, G\rangle = 0$ by (h). Since the projection $\pi_{\mathscr{F}} : \mathscr{M} \to \mathscr{F}$ is a homothety on the fibers, $\ddot{Riem}(V, W)G \in \mathscr{S}(\mathscr{F})$ corresponds to the curvature tensor of each fiber applied to V, W, G. Consequently, $\ddot{Riem}(V, W)G$ and $\bar{Riem}(V, W)G$ are related by the Gauss equation. Combining this with the fact that the shape tensor of the fibers is given by $\eta(V, W) = -\frac{\langle V, W\rangle}{\xi}\nabla\xi$, we obtain property (j). □

Definition 2 Let *Riem* be the Riemannian curvature tensor of N. The Ricci curvature tensor Ric of N is the contraction $C_3^1(Riem) \in \mathscr{I}_2^0(N)$ (where $\mathscr{I}_s^r(N)$ is the set of all tensor fields on N), whose components relative to a coordinate system are $R_{ij} = \sum R^m_{ijm}$.

Because of the symmetries of *Riem* the only nonzero contractions of *Riem* are $\pm Ric$.

Lemma 2 *The Ricci curvature tensor Ric is symmetric, and is given relative to a frame field by:*

$$Ric(\Gamma, \Upsilon) = \sum_m \epsilon_m \langle Riem(\Upsilon, \Gamma) E_m, E_m \rangle, \tag{15}$$

where as usual $\epsilon_m = \langle E_m, E_m \rangle$.

Proof $Ric(\Gamma, \Upsilon) = (C_3^1 Riem)(\Gamma, \Upsilon) = \sum \epsilon_m \langle E_m, Riem(\Gamma, \Upsilon) E_m \rangle$. Symmetry by pairs then gives the required formula and shows that Ric is symmetric. □

If its Ricci tensor is identically zero, N is said to be Ricci flat; A flat manifold is certainly Ricci flat.

Here is the revised proposition in English to avoid plagiarism.

Proposition 3 *In a warped product $\mathscr{M} = \mathscr{B} \times_\xi \mathscr{F}$ with $d = dim(\mathscr{F}) > 1$, let Γ, Υ be horizontal vectors and V, W be vertical vectors. Then, the Ricci tensor $\bar{Ric}$ of $\mathscr{M}$ satisfies the following:*

(k) $\bar{Ric}(\Gamma, \Upsilon) = Ric(\Gamma, \Upsilon) - \frac{d}{\xi} H^\xi(\Gamma, \Upsilon)$,

(l) $\bar{Ric}(\Gamma, V) = 0$,

(m) $\bar{Ric}(V, W) = \ddot{Ric}(V, W) - \langle V, W \rangle \xi^*$,
where $\xi^* = \frac{\Delta \xi}{\xi} + (d-1)\frac{\langle \nabla \xi, \nabla \xi \rangle}{\xi^2}$, *and Ric and* $\ddot{Ric}$ *are the lifts of the Ricci curvature of $\mathscr{B}$ and $\mathscr{F}$, respectively.*

The proof is an exercise in tensor computation, just apply the Ricci formula in *Lemma 2* to a frame field on $\mathscr{M}$ whose vector fields are in $\mathscr{S}(\mathscr{B})$ and $\mathscr{S}(\mathscr{F})$.

Now it is possible to provide the classic definition of Einstein-warped product manifold (for more details see e.g. [5]):

Definition 3 A warped product manifold $(\mathscr{M}, \bar{g}) = (\mathscr{B}, g) \times_\xi (\mathscr{F}, \ddot{g})$, with metric tensor $\bar{g} = g + \xi^2 \ddot{g}$, is Einstein if only if:

$$\bar{Ric} = \lambda \bar{g} \iff \begin{cases} Ric - \frac{d}{\xi} \nabla^2 \xi = \lambda g \\ \ddot{Ric} = \mu \ddot{g} \\ \xi \Delta \xi + (d-1)|\nabla \xi|^2 + \lambda \xi^2 = \mu \end{cases} \tag{16}$$

where λ and μ are constants, d is the dimension of $\mathscr{F}$, $\nabla^2 \xi$, $\Delta \xi$ and $\nabla \xi$ are, respectively, the Hessian, the Laplacian and the gradient of $\mathscr{F}$ for g, with $\xi : (\mathscr{B}) \to \mathbb{R}^+$ a smooth positive function.

By contracting $Ric - \frac{d}{\xi} \nabla^2 \xi = \lambda g$ we have:

$$R\xi^2 - \xi \Delta \xi d = n \xi^2 \lambda \tag{17}$$

where n and R are the dimension and the scalar curvature of $\mathscr{B}$ respectively, and from third equation, considering $d \neq 0$ and $d \neq 1$, we have:

$$\xi \Delta \xi d + d(d-1)|\nabla \xi|^2 + \lambda \xi^2 d = \mu d. \tag{18}$$

3 Definition and Construction of a PNDP-Manifold

In the foregoing section a general introduction to the theory of warped product manifolds was given. In this section we will deal with PNDP-manifolds, special types of Einstein sequential warped product manifolds, introduced by Pigazzini et al. [6], whose base-manifold ($\mathscr{B}$) is a Riemannian product with precise characteristics and whose fiber-manifold ($\mathscr{F}$) is a derived smooth manifold that allows a concept of "virtual" dimensions. For more information about Kuranishi neighborhoods (derived smooth manifolds and obstruction bundle) see [7].

Definition 4 A PNDP-manifold is defined as a warped product manifold $(\mathscr{M}, \bar{g}) = (\mathscr{B}, g) \times_\xi (\mathscr{F}, \ddot{g})$ as in *Definition* 3 (17). The base manifold $(\mathscr{B}, g)$ is a Riemannian (or pseudo-Riemannian) product manifold $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$, where the metric g is given by $g = \sum g_i$. Here, $\mathscr{B}_2$ is an Einstein manifold, meaning it satisfies $Ric_2 = \lambda g_2$, with λ as in *Definition* 3 and the metric of $\mathscr{B}_2$ is g_2. Observe that the dimension of $\mathscr{B}$ is n since dimensions of $\mathscr{B}_1$ and $\mathscr{B}_2$ are n_1 and n_2 respectively.

The function $\xi : \mathscr{B} \to \mathbb{R}^+$ is the warp function expressed as $\xi(\Gamma, \Upsilon) = \xi_1(x) + \xi_2(y)$, where ξ_1 and ξ_2 are functions defined on $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively, or it can be a constant function. The fiber manifold $(\mathscr{F}, \ddot{g})$ is a derived flat Riemannian manifold with negative dimensions m; m is considered to be a virtual interger dimention m. Also mind that by a derived smooth manifold we refer to a smooth Riemannian manifold being augmented by a vector bundle of obstructions.
Specifically, we consider $\mathscr{F}$ as $\mathbb{R}^d$ with orthogonal Cartesian coordinates such that $g_{ij} = -\delta_{ij}$, augmented by a vector bundle of obstructions $\mathscr{E} \to \mathbb{R}^d$, with dimension $m = d - rank(\mathscr{E})$, where $rank(\mathscr{E}) = 2d$. In this scenario, if we consider a Kuranishi neighborhood $(\mathbb{R}^d, \mathscr{E}, S)$, where $\mathbb{R}^d$ is the manifold, $\mathscr{E} \to \mathbb{R}^d$ is the obstruction bundle, and $S : \mathbb{R}^d \to \mathscr{E}$ is the section, then the dimension of the derived smooth manifold $\mathscr{F}$ is $dim(\mathbb{R}^d) - rank(\mathscr{E})$.

Furthermore, in the case where $n > d$ (i.e., $\mathscr{M}$ has a positive "virtual" dimension), we only focus on a PNDP-manifold when $n_1 = d = -m$, so that the "virtual" dimension of $\mathscr{M}$, $dim(\mathscr{M})_{virt}$, matches $dim(\mathscr{B}_2)$. In the special case which $n > d$ and $\mathscr{B}_1$ is also an Einstein manifold, we look at a PNDP-manifold only when $\mathscr{B}_1 = \mathscr{B}_2$.

[**Important Note**: Consider $\mathscr{F} := (\mathbb{R}^d, \mathscr{E}, S)$, and note that the principles of (pseudo-)Riemannian geometry are not directly applicable to the obstruction bundle $\mathscr{E}$. Therefore, all (pseudo-)Riemannian geometric operations are executed exclusively on the base space $\mathbb{R}^d$. Despite this, we treat these operations as if they extend to the structure $\mathscr{F}$. For example, we assume that the Ricci curvature of $\mathscr{F}$ is zero,

mirroring the fact that the Ricci curvature of $\mathbb{R}^d$ is zero. Given that standard (pseudo-)Riemannian geometry holds for the underlying smooth manifold (since it is a conventional manifold), we will apply (pseudo-)Riemannian geometric techniques to the associated fiber manifold $\mathscr{F}$. However, these techniques will be performed on $\mathbb{R}^d$ and assumed to be valid for $\mathscr{F}$, while remaining mindful of the virtual dimensions of $\mathscr{F}$.

In this framework, the tangent space and vector fields correspond to those of $\mathbb{R}^d$. The scalar product of two arbitrary vector fields $\ddot{g}\langle V, W\rangle$ on $\mathscr{F}$ is specified as follows: $g_{ij}v^i w^j = -\delta_{ij}v^i w^j = -(v^i w^i)$.

The analysis utilizes an approach akin to the standard study of Einstein sequential warped product manifolds, such as $(\mathscr{M}_1 \times_h \mathscr{M}_2) \times_{\bar{h}} \mathscr{M}_3$ (see [8, 9]), where $h = 1$, $\mathscr{M}_2$ is an Einstein manifold, and $\mathscr{M}_3$ is a derived smooth manifold with negative "virtual" dimensions. The Riemannian curvature tensor and the Ricci curvature tensor of the entire product manifold can be exhibited as the sum of the corresponding curvature tensors of the individual Riemannian manifolds (see [10]).

Proposition 4 *$\mathscr{B}$-product as $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$, where:*

(i) Ric_i *denote the Ricci tensor of* $\mathscr{B}_i$ *with reference to* g_i*, where* $i = 1, 2$*,*
(ii) $\xi(\Gamma, \Upsilon) = \xi_1(x) + \xi_2(y)$*, is the smooth warping function, and* $\xi_i : \mathscr{B}_i \to \mathbb{R}^+$*,*
(iii) $Hess(\xi) = \sum_i \pi^*_{\mathscr{B}_i} Hess_i(\xi_i)$ *is the Hessian with respect t its individual metric, where* $\pi^*_{\mathscr{B}_i}$ *are the respective pullbacks, (and* $\pi^*_{\mathscr{B}_2} Hess_2(\xi_2) = 0$ *since* $\mathscr{B}_2$ *is Einstein),*
(iv) $\nabla\xi$ *is the gradient (then* $|\nabla\xi|^2 = \sum_i |\nabla_i \xi_i|^2$*), and*
(v) $\Delta\xi = \sum_i \Delta_i \xi_i$ *is the Laplacian, (from (iii) then also* $\Delta_2\xi_2 = 0$*).*

Then the Ricci curvature tensor will be:

$$\begin{cases} \bar{Ric}(\Gamma_i, \Gamma_j) = Ric_1(\Gamma_i, \Gamma_j) - \frac{d}{\xi} Hess_1(\xi_1)(\Gamma_i, \Gamma_j) \\ \bar{Ric}(\Upsilon_i, \Upsilon_j) = Ric_2(\Upsilon_i, \Upsilon_j) \\ \bar{Ric}(G_i, G_j) = \ddot{Ric}(G_i, G_j) - \ddot{g}(G_i, G_j)\xi^* \\ \bar{Ric}(\Gamma_i, \Upsilon_j) = 0 \\ \bar{Ric}(\Gamma_i, G_j) = 0, \\ \bar{Ric}(\Upsilon_i, G_j) = 0, \end{cases} \tag{19}$$

where $\xi^* = \frac{\Delta_1\xi_1}{\xi} + (d-1)\frac{|\nabla\xi|^2}{\xi^2}$*, and* Γ_i*,* Γ_j*,* Υ_i*,* Υ_j*,* G_i*,* G_j *are vector fields on* $\mathscr{B}_1$*,* $\mathscr{B}_2$ *and* $\mathscr{F}$*, respectively.*

Similarly, the proof in this scenario can be considered a straightforward exercise: simply rework Proposition 1 by taking into account the characteristics outlined in Definition 4 for the base manifold $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$ *and the warping function* $\xi(\Gamma, \Upsilon) = \xi_1(x) + \xi_2(y)$*. As a result, the Riemann curvature tensors will be redefined by revisiting Proposition 1, and then, using Lemma 2, we will derive the system given in equation (20).*

In relation to *Definition* 3, we state.

Theorem 1 *A PNDP-manifold is a warped product manifold with derived differential fiber-manifold $\mathscr{F} := (\mathbb{R}^d, \mathscr{E}, S)$, and $dim(\mathscr{F})$ a negative integer, as defined in Definition 4, iff:*

$$\bar{Ric} = \lambda \bar{g} \iff \begin{cases} Ric_1 - \frac{d}{\xi}\pi^*_{\mathscr{B}_1}\nabla_1^2\xi_1 = \lambda g_1 \\ \pi^*_{\mathscr{B}_2}\nabla_2^2\xi_2 = 0 \\ Ric_2 = \lambda g_2 \\ \ddot{Ric} = 0 \\ \xi\Delta_1\xi_1 + (d-1)|\nabla\xi|^2 + \lambda\xi^2 = 0, \end{cases} \tag{20}$$

(since the Ricci curvature of $\mathscr{B}$ is Ric, then $Ric = Ric_1 + Ric_2 = \lambda(g_1 + g_2) + \frac{d}{\xi}\pi^*_{\mathscr{B}_1}\nabla_1^2\xi_1$).

Hence Eqs. (18) and (19), will be exhibited as the following when $n = d$ and $n < d$:

$$\bar{R} = \lambda \bar{n} \iff \begin{cases} R_1\xi - \Delta_1\xi_1 d = n_1\xi\lambda \\ \Delta_2\xi_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ \xi\Delta_1\xi_1 + (d-1)|\nabla\xi|^2 + \lambda\xi^2 = 0. \end{cases} \tag{21}$$

where n_1, n_2, R_1 and R_2 are the dimension and the scalar curvature of $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively. While for $n - d > 0$, as per Definition 4, we must set $d = n_1$, so we have:

$$\bar{R} = \lambda \bar{n} \iff \begin{cases} R_1\xi - \Delta_1\xi_1 n_1 = n_1\xi\lambda \\ \Delta_2\xi_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ \xi\Delta_1\xi_1 + (n_1-1)|\nabla\xi|^2 + \lambda\xi^2 = 0. \end{cases} \tag{22}$$

Proof We utilized the condition indicating that the warped product manifold of the system (20), in Proposition 4, is Einstein. From this we obtain that for a PNDP-manifold the system (17) becomes the general system (21), which splits, according to the "virtual" dimension value of $\mathscr{M}$, into (22) and (23). □

We underline that from first equation of the systems, R_1 cannot depend on points on $\mathscr{B}_2$, so this imposes a condition on $\mathscr{F}$.

Remark 2.1 $d = 1$ is a specific situation where systems (22) and (23) require modification. Indeed, when $d = 1$, system (21) leads to the following expression:

$$\bar{Ric} = \lambda \bar{g} \iff \begin{cases} Ric_1 - \frac{1}{\xi}\pi^*_{\mathscr{B}_1}\nabla_1^2\xi_1 = \lambda g_1 \\ \pi^*_{\mathscr{B}_2}\nabla_2^2\xi_2 = 0 \\ Ric_2 = \lambda g_2 \\ \ddot{Ric} = 0 \\ \xi\Delta_1\xi_1 + \lambda\xi^2 = 0, \end{cases} \tag{23}$$

from which system (22) becomes:

$$\bar{R} = \lambda \bar{n} \iff \begin{cases} R_1\xi - \Delta_1\xi_1 = n_1\xi\lambda \\ \Delta_2\xi_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ \xi\Delta_1\xi_1 + \lambda\xi^2 = 0. \end{cases} \tag{24}$$

and system (23) becomes:

$$\bar{R} = \lambda \bar{n} \iff \begin{cases} R_1\xi - \Delta_1\xi_1 = \xi\lambda \\ \Delta_2\xi_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ \xi\Delta_1\xi_1 + \lambda\xi^2 = 0. \end{cases} \tag{25}$$

In summary, derived geometry is employed to define the fiber manifold, thereby allowing the inclusion of negative "virtual" dimensions on $\mathscr{F}$. Since the fiber manifold $\mathscr{F}$ is treated as $\mathbb{R}^d$ from the perspective of (pseudo-)Riemannian geometry, the traditional framework for constructing warped product manifolds (as seen in [4, 5, 11]) remains applicable. For instance, when considering vertical vector fields G and V as lifts from the vector fields on $\mathscr{F}$, the derivation of the relevant formulas proceeds unchanged. The fields Γ and Υ, which are lifted from $\mathscr{B}$, are horizontal and thus constant on the fibers. Consequently, $V[\Gamma, \Upsilon] = 0$, and the scalar product between a vector field on $\mathscr{B}$ and one on $\mathscr{F}$ is zero (i.e., $\langle \Gamma, V \rangle = 0$).

From a differential geometric perspective, this leads to a specific class of Einstein sequential warped product manifolds $(\mathscr{M}, \bar{g}) = (\mathscr{B}_1 \times \mathscr{B}_2, g_1 + g_2) \times_\xi (\mathbb{R}^d, \ddot{g})$, which simplifies the process of finding a broader range of exact solutions to Einstein's field equations without significantly complicating the calculations compared to the Einstein warped product manifolds with Ricci-flat fibers $(\mathscr{F}; \ddot{g})$. For further reference, see [12], where the authors explore conformal metrics in the context of this special class of Einstein sequential warped product manifolds, demonstrating the existence of a family of $\bar{Ric} > 0$ solutions. These findings could be particularly valuable in the construction of warped product spacetime models with positive curvature and flat fibers.

From the perspective of derived geometry, the resulting PNDP-manifold will have a "virtual" dimension of $n + (d - rank(\mathscr{E}))$.

PNDP-metric: In PNDP-manifolds, which feature fibers with a negative "virtual" dimension, it is important to distinguish their metric from that of a traditional Einstein warped product manifold. To reflect the negative "virtual" dimensions of $\mathscr{F}$, the pseudo-Riemannian metric of the fiber-manifold is represented as $\ddot{g} = \sum(d\psi^i)^2_{(m)}$, where m denotes the negative "virtual" dimension of $\mathscr{F}$.

Consequently, the general metric of a PNDP-manifold can be expressed as:

$$\bar{g} = g - \xi^2 \left(\sum_{i=1}^{d} (d\psi^i)^2 \right)_{(m)} = (g_1 + g_2) + (\xi_1 + \xi_2)^2 \left(\sum_{i=1}^{d} (d\psi^i)^2 \right)_{(m)}.$$

Example 2.1 Simple Example—Consider a flat PNDP-manifold with a positive "virtual" dimension. The manifold $(\mathbb{R}^2 \times \mathbb{R}^2) \times [(\mathbb{R}^2 + \mathscr{E})]$ (where $rank(\mathscr{E}) = 4$) is a $(4 - 2)$-PNDP-manifold that is Ricci-flat. Here, $n + m = n - d = 4 - 2 = 2 > 0$, satisfying system (23) (since $dim(\mathscr{B}_1) = dim(\mathscr{F})$), with constant $\xi = 1$ (both ξ_1 and ξ_2 are constants). With $rank(\mathscr{E}) = 4$, we have $m = -2$. The metric for this manifold is: $ds^2 = dt^2 + dx^2 + dy^2 + dz^2 - (du^2 + dv^2)_{(-2)}$.

4 Interpretation of "Virtual" Dimensions

In this section, we discuss the concept of "virtual" dimensions, highlighting the speculative and practical implications of PNDP-manifolds. Given that m is a negative "virtual" dimension ($m = -d$), the dimension of the PNDP-manifold $(\mathscr{M}, g_{\mathscr{M}})$ becomes "virtual" as well ($dim(\mathscr{M}) = dim(\mathscr{B}) + dim(\mathscr{F}) = dim(\mathscr{B}) + m = dim(\mathscr{B}) - d$, where $dim(\mathscr{B}) = dim(\mathscr{B}_1) + dim(\mathscr{B}_2)$ since $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$). This implies that $dim(\mathscr{M})$ can be "virtual" positive, zero, or negative. Speculatively, this means that the negative "virtual" dimensions of $\mathscr{F}$ interact and effectively cancel out with the positive dimensions of $\mathscr{B}$.

For instance, since $\bar{Ric} = \lambda \bar{g}$, according to the definition of PNDP-manifold (*Definition* 4), the manifold $\mathscr{B}_2$ is also Einstein with the same constant λ as the PNDP-manifold, i.e., $Ric_2 = \lambda g_2$. When the "virtual" dimension $dim(\mathscr{M})_{virt} > 0$, then $dim(M)_{virt} = dim(\mathscr{B}_2)$ (again from *Definition* 4). In this speculative interpretation, a special projection acts as a desuspension, "projecting" $\mathscr{M}$ into $\mathscr{B}_2$. This suggests that the manifold $\mathscr{M}$ "emerges" as the manifold $\mathscr{B}_2$, with the rest of $\mathscr{M}$ being hidden. This supports the idea that the negative "virtual" dimensions of $\mathscr{F}$ cancel out with the positive dimension of $\mathscr{B}_1$.

Additional considerations will be made for cases where $dim(\mathscr{M})_{virt} = 0$ and $dim(\mathscr{M})_{virt} < 0$.

The following are the three types of PNDP-manifolds under consideration:

Type (I) A $(n, -n)$-PNDP manifold, which has an overall zero "virtual" dimension ($dim(\mathscr{M}) = dim(\mathscr{B}) + dim(\mathscr{F}) = n + (-n) = 0$). Speculatively, this can be interpreted as an "invisible" manifold, essentially a "point-like manifold" with zero dimensions and "hidden" dimensions.

Type (II) A $(n, -d)$-PNDP manifold, where n (the dimension of the base-manifold $\mathscr{B}$) differs from d (a positive integer representing the dimension of the underlying manifold $\mathbb{R}^d$ of the fiber-manifold $\mathscr{F}$), such that $dim = n + (-d) > 0$. The distinctive speculative characteristic of this manifold is that it appears as another Einstein-manifold (e.g., the $\mathscr{B}_2$ manifold).

Type (III) Similar to Type II, but with $dim = n + (-d) < 0$. This manifold can be considered, through special projection, as the $||(n-d)||$th desuspension of a point.

Definition 5 In general, given an n-dimensional space X, the suspension ΣX increases the dimension to $n + 1$. Thus, the suspension operation allows for an increase in dimension. The inverse operation, Σ^{-1}, is known as desuspension. Therefore, given an n-dimensional space X, the desuspension $\Sigma^{-1}X$ reduces the dimension to $n - 1$ (see [13]).

As previously mentioned, the interpretation suggests that each negative "virtual" dimension counteracts a positive dimension, canceling them out. We therefore consider the relationship between the "virtual" dimension of a PNDP-manifold and the conventional geometric concept of dimension as a desuspension interpreted through special projection.

With $\mathscr{B} = (\mathscr{B}_1 \times \mathscr{B}_2)$ and $\mathscr{F} := (\mathbb{R}^d, \mathscr{E}, S)$, the PNDP-manifold is given by $PNDP = \mathscr{B} \times_\xi \mathscr{F} = (\mathscr{B}_1 \times \mathscr{B}_2) \times_{(\xi_1+\xi_2)} \mathscr{F}$, with $dim(\mathscr{B}) - dim(\mathbb{R}^d) = dim(PNDP) = n - d$, leading to the following scenarios:

Type (I) If $(n - d) = 0$ (i.e., (22)), the projection is:

$$\tau_{(0)} : PNDP \to \text{point-like manifold},$$

Type (II) If $(n - d) > 0$ (i.e., (23)), the projection is:

$$\tau_{(>0)} : PNDP \to \mathscr{B}_2,$$

Type (III) If $(n - d) < 0$ (i.e., (22)), the projection is:

$$\tau_{(<0)} : PNDP \to \Sigma^{n-d}(p),$$

where $\Sigma^{n-d}(p)$ represents the $||(n-d)||$th desuspension of a point.

Example 2.2 Speculative Example—Consider a (8 – 4)-PNDP-manifold $\mathscr{M} = (\mathscr{B}_1 \times \mathscr{B}_2) \times_\xi \mathscr{F}$ with non-constant ξ. Since $n + m = n - d = 8 - 4 = 4 > 0$, then (from *Definition* 4) $\mathscr{M}$ is considered a PNDP-manifold only if $dim(\mathscr{B}_1) =$

$-dim(\mathscr{F})$, so $dim(\mathscr{B}_1) = dim(\mathbb{R}^4)$, as $m = -4 = d - rank(\mathscr{E})$, where $rank(\mathscr{E}) = 2d$. Therefore, the PNDP-manifold satisfies system (23), and, as per *Definition* 4, $\mathscr{B}_2$ is an Einstein-manifold, i.e., $Ric_2 = \lambda g_2$. Thus, we have:
$\tau_4 : (8-4)$-PNDP $\rightarrow \mathscr{B}_2$. Under this interpretation of "virtual" dimensions, the $(8-4)$-PNDP manifold is, from a speculative point of view, identified with the Einstein-manifold $\mathscr{B}_2$.

Example 2.3 Speculative Example—Consider a $(6-6)$-PNDP-manifold $(\mathscr{B}_1 \times \mathscr{B}_2) \times_\xi \mathscr{F}$ with non-constant ξ. Since $n + m = n - d = 6 - 6 = 0$, then $dim(\mathscr{B}) = dim(\mathscr{B}_1) + dim(\mathscr{B}_2)$ must equal $-dim(\mathscr{F})$. Therefore, this PNDP-manifold satisfies system (22), and we have:
$\tau_0 : (6-6)$-PNDP $\rightarrow$ point-like manifold (zero-dimension). For this case, $rank(\mathscr{E}) = 12$.

Example 2.4 Speculative Example—Consider a $(6-8)$-PNDP-manifold $(\mathscr{B}_1 \times \mathscr{B}_2) \times_\xi \mathscr{F}$ with non-constant ξ. Since $n + m = n - d = 6 - 8 = -2 < 0$, this PNDP-manifold also satisfies system (22), and we have: $\tau_{-2} : (6-8)$-PNDP $\rightarrow \Sigma^{-2}(p)$ (double desuspension of a point, i.e., a -2-dimensional manifold). In this case, $rank(\mathscr{E}) = 16$.

Example 2.5 Speculative Example using Trivial Example—Considering the manifold of the *Trivial Example*, where we have the special case $\mathscr{B}_1 = \mathscr{B}_2$, that is $\mathbb{R}^2$, the desuspension/projection will be: $\tau_2 : (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 + \mathscr{E}) \rightarrow \mathbb{R}^2$, i.e., we identify the $(4-2)$-PNDP manifold with $\mathbb{R}^2$ (Fig. 1).

More specifically, the negative virtual dimensions of $\mathscr{F}$ can be seen as representations of internal structures of the manifold that do not manifest themselves as traversable spatial directions, but rather as components that influence the algebraic structure of space. These algebraic components act on more abstract levels, modifying the behavior of physical quantities, but without corresponding to directly accessible spatial directions. In a quantum context, negative virtual dimensions can represent perturbations in the quantum vacuum state. These perturbations do not generate traversable spatial directions, but influence the geometry through correction terms. In this sense, they contribute to a sort of "cancellation" or "crushing" of the traversable spatial dimensions, analogously to how quantum fluctuations can reduce the available energy spectrum through the annihilation of particle-antiparticle pairs. The process described leads to a reduction of the effective dimensionality of the emerging manifold. This means that the perceivable and traversable spatial dimensions are effectively fewer than the theoretical ones of the total manifold.

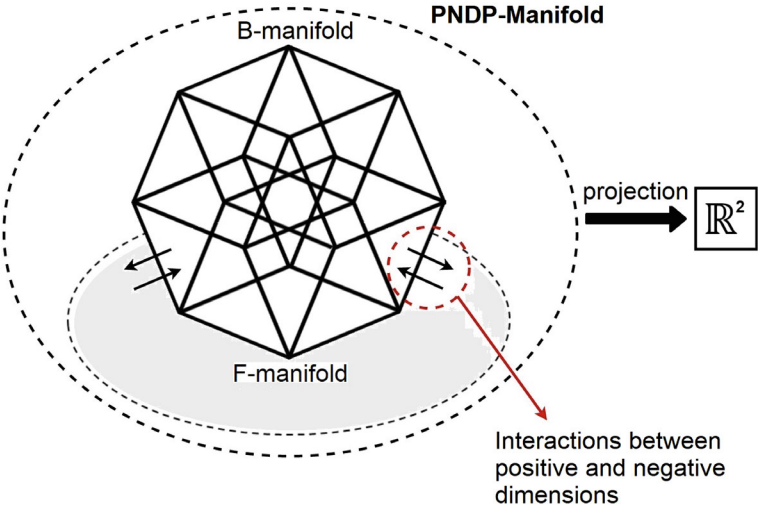

Fig. 1 The figure represents the *Trivial Example* according to the speculative approach to "virtual" dimensions described in the section. The PNDP-maifold is $(\mathbb{R}^2 \times \mathbb{R}^2) \times [(\mathbb{R}^2 + \mathscr{E})]$ (where the base-manifld is represented in the figure as a tesseratto) with $rank(\mathscr{E}) = 4$, and following the approach about the interaction between dimensions (in which the negative "virtual" dimensions of $\mathscr{F}$ interact canceling/hiding with the dimensions of $\mathscr{B}_1$), the manifold resulting from the projection will be $\mathbb{R}^2$

5 Possible Multi-disciplinary Applications

In a PNDP-manifold, the concept of dimension deviates from traditional geometric interpretations, instead representing "virtual" dimensions that offer a speculative lens through which reality might be perceived. When we define a particular algebra on a manifold, the dimension corresponding to that algebra may differ from the apparent dimension of the underlying manifold. This "virtual" dimension becomes relevant depending on the mathematical framework we choose to observe. Thus, in the context of PNDP-manifolds, varying perspectives on the manifold can yield different dimensional interpretations.

This suggests that nature might manifest in ways that aren't immediately aligned with our observations. The PNDP-manifold could serve as a tool to uncover "emergent" dimensions-revealing aspects of reality that are not directly observable. This framework allows for the exploration of "hidden" dimensions, offering a new way to interpret dimensionality beyond our conventional understanding.

The PNDP-manifold, as discussed in detail in earlier sections (see Sects. 3 and 4), has potential applications in various fields. For instance, it has been speculatively applied in contexts such as string theory, D-branes, and discrete gravity theory, as explored in studies like [14–16] (Fig. 2).

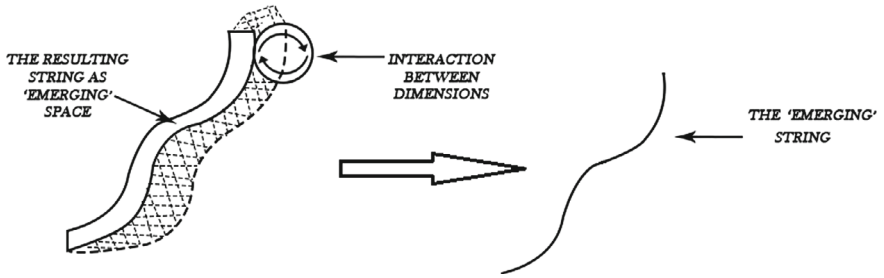

Fig. 2 A $2-1$- PNDP is shown,i.e. $(I_1 \times I_2) \times (I_3 + E)$. A line interval emerges from the interaction between positive and virtual negative dimensions which is topologically equivalent to a string

This perspective is grounded in the idea that some dimensions of space-time, due to their intrinsic properties or initial conditions, might be better described mathematically as "virtual" negatives. This approach implies that these dimensions interact with the other dimensions present in the configuration. It's conceivable that space itself might be a secondary phenomenon, arising from more fundamental forces. In such a scenario, certain dimensions could disappear, as seen, for example, in how the extra dimensions which are not gravitational can be dynamically generated by four-dimensional gauge theories. A negative "virtual" dimension might therefore represent a dimension where particular fundamental forces allow for the emergence of other dimensions.

A similar concept can be applied in econophysics, such as in the modeling of financial markets, where phenomena like dark volatility, akin to ghost fields, influence the market dynamics.

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A Fibonacci Sequence Derived from Cardinalities of the Set of Open Sets of Khalimsky Topological Subspaces

P. Gnanachandra, Sang-Eon Han, and Saeid Jafari

Abstract In this chapter we present some properties of the Khalimsky topological line. Moreover, we derive a Fibonacci sequence from cardinalities of the set of open sets of some Khalimsky topological subspaces.

Keywords Digital topology · Khalimsky line · Alexandroff topology · Finite T_0 spaces

Mathematics Subject Classification 54A05 · 54J05 · 68U05

1 Introduction

Digital topology is a term that has arisen for the study of geometric and topological properties of digital images. Topological properties of images on Cathode ray tube are very important in a wide range of applications, namely, computer graphics, pattern analysis, robotic design, etc. There are two approaches to study digital images.

1. Graph theoretic approach
2. Topological approach

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The first approach was initiated by A. Rosenfeld in the late 1960s. The topological approach was initiated by Kong, Kopperman, Meyer and Khalimsky and others in 1990s. The two approaches are coextensive for finite spaces. The study of images begins with Jordan Curve Theorem which states that a simple closed curve separates the real plane $\mathbb{R} \times \mathbb{R}$ into two connected components. We want an analogue for the plane $\mathbb{Z} \times \mathbb{Z}$. For this we require a connected topology on $\mathbb{Z}$ regarded as an ordered set. Hence we are primarily interested in finite T_0 spaces defined on the intervals of $\mathbb{Z}$. Since we need a connected topology on the set of lattice points of the computer screen, the topologies on the finite sets are found to be a more suitable avenue in the study of digital topology. Khalimsky et al. utilized a connected topology on a finite ordered set to computer graphics. One such topology on $\mathbb{Z}$ is the topology generated by the triplets $\{2a-1, 2a, 2a+1\} : a \in \mathbb{Z}$ as subbase. As this topology was introduced by Khalimsky it is also called Khalimsky topology or digital line.

2 Various Notions Related to the Study of the Khalimsky Topological Line

A space (X, τ) is door space if every subset is either open or closed, an Alexandroff space if every intersection of open set is open, an extremely disconnected (briefly ED) if the closure of every open set is open and is locally finite if each point lies in a finite open set and in a finite closed set. In a topological space X, a subset B of X is pointwise dense if $\cup\{Cl_\tau(\{x\}) : x \in B$ and $\{x\}$ is open $\} = X$. For $A \subseteq X$, the closure and the interior of A are denoted by $Cl(A)$, $Int(A)$, respectively. B is said to be α-open (semi-open preopen and semi-preopen, respectively) if $B \subseteq Int_\tau(Cl_\tau(Int_\tau(B)))$ ($B \subseteq Cl_\tau(Int_\tau(B))$, $B \subseteq Int_\tau(Cl_\tau(B))$ and $A \subseteq Cl_\tau(Int_\tau(Cl_\tau(B)))$, $respectively$) and is α-closed (semi-closed, preclosed and semi-preclosed, respectively) if $Cl_\tau(Int_\tau(B)) \subseteq B$ ($Int_\tau(Cl_\tau(B)) \subseteq B$, $Cl_\tau(Int_\tau(B)) \subseteq B$ and $Int_\tau(Cl_\tau(Int_\tau(B)))$, respectively). The α-closure of a set B, denoted by $\alpha Cl_\tau(B)$ with respect to topology τ is the intersection of all α-closed sets containing B. The semi-closure of B denoted by $sCl_\tau(B)$, the preclosure of B denoted by $pCl_\tau(B)$ and the semi-preclosure of B denoted by $spCl_\tau(B)$ can be analogously defined. A space (X, τ) is R_0 if $Cl_\tau(\{x\}) \subseteq U$, whenever $x \in U$ and U is open in τ. We denote the set of all α-open sets in τ by τ^α and a set consisting of all α-open sets of some topology is called an α-structure on X. In X, arbitrary intersection of open sets is G_δ-set and arbitrary union of closed sets if F_σ-set.

3 Some Properties of the Khalimsky Topological Line

In this section, we consider a topology on $\mathbb{Z}$, the set of all integers. This topology plays an important role in computer graphics. We also study the concepts of digital

line and intervals. Let D denote the set of all triplets $\{2a-1, 2a, 2a+1\}, a \in \mathbb{Z}$. Then D will be a subbase for some topology on $\mathbb{Z}$. This topology is called the Khalimsky topology on $\mathbb{Z}$, denoted by κ. Then $\mathbb{Z}$ together with this topology is called the digital line. Clearly a base for this topology is $D \cup \mathfrak{B}$, where $\mathfrak{B} = \{\{n\} : n \in \mathbb{Z}$ and n is odd$\}$. This follows from the following observation.

Let $A = \{2a-1, 2a, 2a+1\}$ and $B = \{2b-1, 2b, 2b+1\}$ where a and b are integers. Then

$$A \cap B = \begin{Bmatrix} A & \text{if} & a = b \\ \{2a+1\} & \text{if} & a = b-1 \\ \{2a-1\} & \text{if} & a = b+1 \\ \emptyset & \text{otherwise} & \end{Bmatrix} \tag{1}$$

Let A be a subset of $\mathbb{Z}$. Then A is open if and only if for every x in A the following holds: x is odd or x is even with $x-1, x+1 \in A$. A is closed if and only if for every x in A the following holds: x is even or x is odd with $x-1, x+1 \in A$.
Let $N(x)$ denote the smallest neighborhood of x in $(\mathbb{Z}, \kappa)$. Then

$$N(x) = \begin{Bmatrix} \{x\} & \text{if} & x \text{ is odd} \\ \{x-1, x, x+1\} & \text{if} & x \text{ is even} \end{Bmatrix} \tag{2}$$

Let $N[x]$ denote the smallest closed set containing x. Then

$$N[x] = \begin{Bmatrix} \{x\} & \text{if} & x \text{ is even} \\ \{x-1, x, x+1\} & \text{if} & x \text{ is odd} \end{Bmatrix} \tag{3}$$

Using (2) and (3), one can easily verify the following results.

- $Cl_\kappa(\{2a+1\}) = \{2a, 2a+1, 2a+2\}$
- $Int_\kappa(Cl_\kappa\{2a+1\}) = Int_\kappa\{2a, 2a+1, 2a+2\} = \{2a+1\}$
- $Cl_\kappa(Int_\kappa(\{2a+1\})) = Cl_\kappa(\{2a+1\}) = \{2a, 2a+1, 2a+2\}$
- $Int_\kappa(Cl_\kappa(Int_\kappa(\{2a+1\}))) = \{2a+1\}$
- $Cl_\kappa(Int_\kappa(Cl_\kappa(\{2a+1\}))) = \{2a, 2a+1, 2a+2\}$
- $Int_\kappa(Cl_\kappa(\{2a\})) = \emptyset$ and $Cl_\kappa(Int_\kappa(\{2a\})) = \emptyset$.
- $Cl_\kappa(\{2a, 2a+1\}) = \{2a, 2a+1, 2a+2\}$.
- $Int_\kappa(Cl_\kappa(\{2a, 2a+1\})) = \{2a+1\}$
- $Cl_\kappa(Int_\kappa(\{2a, 2a+1\})) = \{2a, 2a+1, 2a+2\}$.

As an implication of the above relations, we have the following results.

- $sCl_\kappa(\{2a+1\}) = \{2a+1\}$.
- $\alpha Cl_\kappa(\{2a+1\}) = \{2a, 2a+1, 2a+2\} = Cl_\kappa(\{2a+1\})$.
- $pCl_\kappa(\{2a+1\}) = \{2a, 2a+1, 2a+2\} = Cl_k(\{2a+1\})$
- $spCl_\kappa(\{2a+1\}) = \{2a+1\}$.

Han and Na [3] proved that the set of all odd integers is dense in $(\mathbb{Z}, \kappa)$. Based on this work, we now obtain the following:

Proposition 1 *In* $(\mathbb{Z}, \kappa)$,

(i) Every dense subset is open,
(ii) The set of all odd integers is pointwise dense,
(iii) Every F_σ *set is closed,*
(iv) Every G_δ *set is open,*
(v) If $A \subseteq 2\mathbb{Z}$ *and* A *is finite, then* A *is closed.*

Proof

(i) Let B be a dense subset of $\mathbb{Z}$. Then $B = O \cup A$ where A is a subset of even integers and O is a set of all odd integers. To prove that B is open it is enough to prove that B is open where $|A| = 1$. Suppose $A = \{x\}$. Since x is even $B = O \cup \{x - 1, x, x + 1\}$ and hence B is open.
(ii) Let A denote the set of all odd integers. Then A is open in $(\mathbb{Z}, \kappa)$. For each integer x, $Cl_\kappa(x) = \{x - 1, x, x + 1\}$. This proves that $Z = \cup_{x \in A} Cl_\kappa(x)$. Hence A is pointwise dense in $(\mathbb{Z}, \kappa)$.
(iii) Suppose $A = \cup A_n$ where each A_n is closed in $(\mathbb{Z}, \kappa)$. Let $x \in A$. Then $x \in A_n$ for some n. If x is even, we are done. If x is odd then $\{x - 1, x, x + 1\}$ is contained in A_n and hence contained in A.
(iv) Suppose $A = \cap A_n$ where each A_n is open in $(\mathbb{Z}, \kappa)$. Let $x \in A$. Then $x \in A_n$ for every n. If x is odd then $\{x\}$ is an open set contained in A_n. If x is even then $\{x - 1, x, x + 1\}$ is an open set contained in A_n and hence contained in A. This proves that A is open.
(v) Follows from the fact that $\{2a\}$ is closed for every integer n.

□

It is clear that $(\mathbb{Z}, \kappa)$ is T_0 but not T_1. Let us now consider the following.

Proposition 2 *The space* $(\mathbb{Z}, \kappa)$ *is*

(i) Not an R_0,
(ii) Not a door space,
(iii) Not an ED,
(iv) Locally finite,
(v) Connected.

Proof

(i) Since $Cl_\kappa(\{2a - 1\}) = \{2a, 2a - 1, 2a + 1\}$ and $\{2a - 1\}$ is open, it follows that $(\mathbb{Z}, \kappa)$ is not R_0.
(ii) Since $\{2a, 2a - 1\}$ is neither open nor closed, $(\mathbb{Z}, \kappa)$ is not a door space.
(iii) Since $\{2a + 1\}$ is open and $Cl_\kappa(2a + 1) = \{2a, 2a + 1, 2a + 2\}$ is not open, it follows that $(\mathbb{Z}, \kappa)$ is not extremely disconnected.

(iv) Let x be point in $\mathbb{Z}$. If x is odd then $\{x\}$ is a finite open set containing x and $\{x-1, x, x+1\}$ is a finite closed set containing x. If x is even then $\{x-1, x, x+1\}$ is a finite open set containing x and $\{x\}$ is a finite closed set containing x. This proves that $(\mathbb{Z}, \kappa)$ is a locally finite space. Moreover, since every locally finite space is Alexandroff, therefore $(\mathbb{Z}, \kappa)$ is Alexandroff.

(v) Let $A \neq \emptyset$ be both open and closed in $(\mathbb{Z}, \kappa)$. Fix $x \in A$.
Case 1. Assume that x is odd. Since A is closed $x-2 \in A$ and $x+1 \in A$. Therefore $\{x-1, x, x+1\} \subseteq A$. Since $x-1$ and $x+1$ are both even and A is open, thus $\{x-2, x-1, x, x+1, x+2\} \subseteq A$. Proceeding by the same token, we see that $\mathbb{Z} = A$.
Case 2. Assume that x is even. Since A is open $x-1 \in A$ and $x+1 \in A$. Therefore $\{x-1, x, x+1\} \subseteq A$. Since $x-1$ and $x+1$ are both odd and since A is closed, $\{x-2, x-1, x, x+1, x+2\} \subseteq A$. Proceeding by the same token, we see that $\mathbb{Z} = A$. This proves the proposition.

4 A Fibonacci Sequence Derived from Cardinalities of the Set of Open Sets of Some Khalimsky Topological Subspaces

In this section, we study the subspaces of $(\mathbb{Z}, \kappa)$ and investigate the behavior of cardinalities of certain subspace topologies. In fact we prove here that the cardinalities of topologies on the digital intervals

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots \{1, 2, 3, \ldots, n\}, \ldots$$

form a subsequence of the Fibonacci sequence. We also give some conjectures on the digital interval $\{1, 2, 3, \ldots, n\}$. Let $E = \{1, 2, 3, 4, 5\}$. Then $\kappa_E = T_5$, the subspace topology generated by $\{\{1\}, \{5\}, \{1, 2, 3\}, \{3, 4, 5\}\}$. Hence

$$T_5 := \left\{ \begin{array}{l} \emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 2, 3\}, \{1, 3, 5\}, \\ \{3, 4, 5\}, \{1, 2, 3, 5\}, \{1, 3, 4, 5\}, E\} \text{ (see Appendix).} \end{array} \right\}$$

Then (E, T_5) is a subspace of $(\mathbb{Z}, \kappa)$.

Levine defined that a set B has the property Q if the interior and the closure operators commute on A and characterized the sets having the property Q. More precisely, in a topological space (X, τ), a set B has the property Q if $Int_\tau(Cl_\tau(B)) = Cl_\tau(Int_\tau(B))$. B is a p-set if $Cl_\tau(Int_\tau(B)) \subseteq Int_\tau(Cl_\tau(B))$ and q-set if $Int_\tau(Cl_\tau(B)) \subseteq Cl_\tau(Int_\tau(B))$. $q(\tau)$, $p(\tau)$, and $Q(\tau)$ denote the collection of all q-sets, p-sets, and sets satisfying Q property in τ.

The next four propositions can be easily proved by using the Appendix.

Proposition 3 *The topology T_5 is an α-structure on $E = \{1, 2, 3, 4, 5\}$, that is, $T_5^\alpha = T_5$.*

Proposition 4 *$q(T_5)$ is the discrete topology on E.*

Proposition 5 *$p(T_5)$ is the indiscrete topology on E.*

Proposition 6 *$Q(T_5)$ is a topology on E other than discrete topology and the indiscrete topology on E.*

Let $T_1, T_2, T_3, T_4, T_5, T_5$, and T_7 be the topologies on the digital intervals $[1,1]\cap\mathbb{Z}$, $[1,2]\cap\mathbb{Z}$, $[1,3]\cap\mathbb{Z}$, $[1,4]\cap\mathbb{Z}$, $[1,5]\cap\mathbb{Z}$, $[1,6]\cap\mathbb{Z}$ and $[1,7]\cap\mathbb{Z}$ respectively. Then from the Appendix it can be proved that

$|T_1| = 2$
$|T_2| = 3$
$|T_3| = 5$
$|T_4| = 8$
$|T_5| = 13$
$|T_6| = 21$ *and*
$|T_7| = 34$.

The above results motivate us to prove the following theorem.

Theorem 1 *If T_{n-1}, T_n, T_{n+1} are the topologies on the digital intervals $[1, n-1]\cap\mathbb{Z}$, $[1,n]\cap\mathbb{Z}$ and $[1,n+1]\cap\mathbb{Z}$ respectively, inherited from the Khalimsky topology κ on $\mathbb{Z}$, then $|T_{n+1}| = |T_n| + |T_{n-1}|$.*

Proof

- **Case 1**. Assume that $n+1$ is odd. Since $[1,n-1]\cap\mathbb{Z} \subseteq [1,n+1]\cap\mathbb{Z}$ and $n-1$ is odd, it is easy to see that $T_{n-1} \subseteq T_{n+1}$. Now $\{n+1\}$ is a basic open set in T_{n+1}. Let $T = \{A \cup \{n+1\} : A \in T_n\}$. Then it can be proved that $T_{n+1} = T_{n-1} \cup T$ and $T \cap T_{n-1} = \emptyset$. This proves that $|T_{n+1}| = |T_n| + |T_{n-1}|$.
- **Case 2**. Assume that $n+1$ is even. Since $[1,n]\cap\mathbb{Z} \subseteq [1,n+1]\cap\mathbb{Z}$ and n is odd, we clearly obtain $T_n \subseteq T_{n+1}$. Now, $\{n, n+1\}$ is a basic open set in T_{n+1}. Let $T = \{A \cup \{n, n+1\} : A \in T_{n-1}\}$. Then it can be proved that $T_{n+1} = T_n \cup T$ and $T \cap T_n = \emptyset$. This proves that $|T_{n+1}| = |T_n| + |T_{n-1}|$. This completes the proof.□

We observe that Propositions 3, 4, 5, and 6 hold for the topology T_5. We can also verify that these propositions hold for the topologies T_1, T_2, T_3, T_4, T_6, and T_7. These ideas motivate us to make the following conjectures. Let T_n be the subspace topology on the digital interval $[1,n]\cap\mathbb{Z}$, inherited from the Khalimsky topology κ on $\mathbb{Z}$. Then

Conjecture 1 $T_n^\alpha = T_n$

Conjecture 2 *$q(T_n)$ is the discrete topology*

Conjecture 3 *$p(T_n)$ is the discrete topology*

Conjecture 4 *$Q(T_n)$ is the topology on $[1,n]\cap\mathbb{Z}$ other than the discrete topology and the indiscrete topology.*

Appendix

- $E = \{1\}$, $T_1 = \{\emptyset, E\}$, and $|T_1| = 2$.
- $E = \{1, 2\}$, $T_2 = \{\emptyset, 1, E\}$, and $|T_2| = 3$.
- $E = \{1, 2, 3\}$, Subbase = {1, 123, 3}, Base = {1, 3, E}, $T_3 = \{\emptyset, 1, 3, 13, E\}$, and $|T_3| = 5$.
- $E = \{1, 2, 3, 4\}$, Subbase = {1, 123, 34}, Base = {1, 3, 123, 34}, $T_4 = \{\emptyset, 1, 3, 13, 34, 123, 134, E\}$, and $T_4 = 8$.
- $E = \{1, 2, 3, 4, 5, 6\}$, Subbase = {1, 123, 345, 56}, Base = {1, 3, 123, 345, 56}, $T_6 = \{\emptyset, 1, 3, 5, 13, 15, 35, 56, 123, 345, 135, 156, 356, 1235, 1345, 1356, 3456, 12356, 12345, 13456, E\}$, and $|T_6| = 21$.
- $E = \{1, 2, 3, 4, 5, 6, 7\}$, Subbase = {1, 123, 345, 567, 7}, Base = {1, 3, 5, 7, 123, 345, 567}, and $T_7 = \{\emptyset, 1, 3, 5, 7, 13, 15, 17, 35, 37, 57, 123, 345, 567, 135, 137, 157, 357, 1235, 1345, 1237, 1567, 1357, 3457, 3567, 12345, 12357, 13567, 34567, 13457, 123457, 123567, 134567, E\}$, and $|T_7| = 34$.
- $E = \{1, 2, 3, 4, 5\}$, Subbase = {1, 123, 345, 5}, Base = {1, 3, 5, 123, 345}, and $T_5 = \{\emptyset, 1, 3, 5, 13, 15, 135, 123, 345, 1235, 1345, E\}$.

A	Int (A)	Cl (A)	Cl(Int(A))	Int(Cl(A))	Int(Cl(Int(A)))	Cl(Int(Cl(A)))
1	1	12	12	1	1	12
2	∅	2	∅	∅	∅	∅
3	3	234	234	3	3	234
4	∅	4	∅	∅	∅	∅
5	5	45	45	5	5	45
12	1	12	12	1	1	12
13	13	1234	1234	123	123	1234
14	1	124	12	1	1	12
15	15	1245	1245	15	15	1245
23	3	234	234	3	3	234
24	∅	24	∅	∅	∅	∅
25	5	245	45	5	5	45
34	3	234	234	3	3	234
35	35	2345	2345	345	345	2345
45	5	45	45	5	5	45
123	123	1234	1234	123	123	1234
124	1	124	12	1	1	12
125	15	1245	1245	15	15	1245
134	13	1234	1234	123	123	1234
135	135	E	E	E	E	E
145	15	1245	1245	15	15	1245
234	3	234	234	3	3	234
235	35	2345	2345	345	345	2345
245	5	245	45	5	5	45
345	345	2345	2345	345	345	2345
1234	123	1234	1234	123	123	1234
1235	1235	E	E	E	E	E
1245	15	1245	1245	15	15	1245
1345	1345	E	E	E	E	E
2345	345	2345	2345	345	345	2345

$T_5^{\alpha} = T_5$; $q(T_5) =$ The power set of E; $p(T_5) = \{\emptyset, E\}$; $Q(T_5) = \{\emptyset, 4, 24, 135, 1235, 1345, E\}$ and $|T_5| = 13$.

Conclusion

We investigated the properties of the topological space $(\mathbb{Z}, \kappa)$ where κ is the Khalimsky topology. We further discussed the subspaces, namely, digital intervals of the digital line $(\mathbb{Z}, \kappa)$.

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A Type of Compactness in Neutrosophic Topological Space via Grills

Runu Dhar and Gour Pal

Abstract We would study compactness via grills in a neutrosophic topological space. We call it neutrosophic weak $\mathscr{G}$-compactness. We would investigate some basic properties and characterization theorems of it. We would also obtain its relationship with other known neutrosophic compactness. Lastly, we would exhibit and study neutrosophic weak compactness through neutrosophic θ-open sets.

1 Introduction

Zadeh [20] initiated the notion of fuzzy set in order to solve real-life phenomena due to uncertainty. Thereafter, intuitionistic fuzzy set was invented by Atanassov [1] associating truth and falsity together. But it was not sufficient to address all real-life problems in an ambiguous environment. In order to solve real-life problems on decision-making under uncertainty, Smarandache [17] revealed neutrosophic set associating with three independent mappings, such as truthness (f), falsity (g) and indeterminacy (h) of the universal set X. These three functions are completely independent. Smarandache [18] further investigated the applications of the neutrosophic theory. Thereafter, Salama and Alblowi [14, 15] introduced and investigated neutrosophic topological space. The brilliant notion of a grill was initiated by Choquet [2]. Compactness in topological space via grills was initiated by Roy and Mukherjee [12]. The notion of grill in neutrosophic topological space and neutrosophic minimal space was introduced by Pal et al. [10]. Pal and Dhar [9] introduced the notion of compactness in neutrosophic minimal space. Karthik and Arockiarani [6, 7] investigated weak compactness by grills in general topology. Besides them, many researchers [3, 4, 11, 16, 19] investigated neutrosophic topological space. Such works motivated

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us to introduce and investigate basic properties and results of weak compactness via grills. Throughout this paper, we call it neutrosophic weak $\mathscr{G}$-compactness. We make sections of the article as mention below. The next section briefly focuses on known definitions and results related to the article. Next, we study neutrosophic weak $\mathscr{G}$-compact space. We also investigate some basic properties and theorems of this space. Section 4 reveals the neutrosophic weak $\mathscr{G}$-compact space through neutrosophic θ-open sets and focuses its basic properties.

2 Preliminaries

Here we mention a few basic definitions and results.

Definition 1 ([2]) Let X be a non-empty set, and let $\mathscr{G}$ be a non-empty collection of subsets of X. Consider $A, B \subseteq X$. The collection $\mathscr{G}$ is called a *grill* of X if it satisfies the following conditions:

(i) If $A \in \mathscr{G}$ and $A \subseteq B$, then $B \in \mathscr{G}$.
(ii) If $A \cup B \in \mathscr{G}$, then either $A \in \mathscr{G}$ or $B \in \mathscr{G}$.

Definition 2 ([12]) Let $\mathscr{G}$ be a grill on a topological space (X, τ). A cover $\{V_i : i \in J\}$ of X is called a $\mathscr{G}$-cover if there exists a finite subset $J_0 \subseteq J$ such that

$$X \setminus \bigcup_{i \in J_0} V_i \notin \mathscr{G}.$$

Definition 3 ([13]) Consider a non-empty set X, a grill $\mathscr{G}$ on it, and $U = \{U_n : n \in \mathbb{N}\}$ is a countable subset of it. Hence, $\mathscr{G}$ is termed a σ-grill if $U \notin \mathscr{G}$ whenever $U_n \notin \mathscr{G}$ for each $n \in \mathbb{N}$.

Definition 4 ([14]) A *neutrosophic set* (NS, in short) F in a universal set W is a set where each element is associated with truth, falsity, and indeterminacy membership values, defined by three independent functions f_F, g_F, and h_F, which take values in the interval [0, 1]. The neutrosophic set F is defined as

$$F = \{(x, f_F(x), g_F(x), h_F(x)) : x \in W\},$$

where $f_F(x)$, $g_F(x)$, and $h_F(x)$ are the truth, falsity, and indeterminacy membership values, respectively. Each of these functions satisfies the following condition:

$$0 \leq f_F(x) + g_F(x) + h_F(x) \leq 3 \quad \text{for all } x \in W.$$

Definition 5 ([14]) We denote the *empty neutrosophic set* and the *whole neutrosophic set* on W as 0_η and 1_η, respectively. These are defined as follows:

$$\text{(i)} \quad 0_\eta = \{(x, 0, 1, 1) : x \in W\},$$
$$\text{(ii)} \quad 1_\eta = \{(x, 1, 0, 0) : x \in W\}.$$

Definition 6 ([14]) Consider ρ as a collection of neutrosophic sets of a non-empty set W. ρ is said to be a *neutrosophic topology* of W if the following properties are satisfied:

(i) The empty neutrosophic set 0_η and the entire neutrosophic set 1_η are members of ρ.
(ii) ρ is closed with respect to finite intersections.
(iii) ρ is closed with respect to arbitrary unions.

Here, (W, ρ) is called a neutrosophic topological space of W. The elements of ρ are referred to as neutrosophic open sets, and their complements are called neutrosophic closed sets.

Throughout this article, (W, ρ) denotes a neutrosophic topological space unless otherwise noted.

Definition 7 ([14]) Consider V as a neutrosophic set in (W, ρ). The *neutrosophic interior* and *neutrosophic closure*, denoted by $N_{\text{int}}(V)$ and $N_{\text{cl}}(V)$, respectively, are defined as follows:

$$N_{\text{int}}(V) = \bigcup \{F : F \text{ is a neutrosophic open set in } W \text{ and } F \subseteq V\},$$

$$N_{\text{cl}}(V) = \bigcap \{H : H \text{ is a neutrosophic closed set in } W \text{ and } V \subseteq H\}.$$

Remark 1 [14] Evidently, $N_{cl}(V)$ $N_{int}(V)$ is the smallest (largest) neutrosophic closed (open) set on W.

Proposition 1 *([14]) For any neutrosophic set V in (W, ρ), the following hold:*

(i) $N_{int}(V^c) = N_{cl}(V)^c$,
(ii) $N_{cl}(V^c) = N_{int}(V)^c$.

Definition 8 ([10]) Consider U a non-null set. A subcollection $\mathcal{G}$ of neutrosophic sets on U (not containing 0_η) and $F, H \subseteq U$ is called a grill of U when $\mathcal{G}$ obeys the following axioms:

- $F \in \mathcal{G}$, $H \subseteq B$ implies $H \in \mathcal{G}$,
- $F \cup H \in \mathcal{G}$ implies $F \in \mathcal{G}$ or $H \in \mathcal{G}$.

Definition 9 ([8]) A neutrosophic point $u_{(\alpha,\beta,\gamma)}$ is called neutrosophic θ-cluster point of a neutrosophic set V if and only if for each U in ϵ-q-neighbourhood of $u_{(\alpha,\beta,\gamma)}$, $\mathrm{Ncl}(U)qV$ and collection of all them is called neutrosophic θ-closure of V, denoted as Ncl_θ.

A neutrosophic set V is neutrosophic θ-closed (NθCS in brief) if and only if $V = \mathrm{Ncl}_\theta(V)$ and its complement is neutrosophic θ-open set (in brief, NθOS).

Definition 10 ([5]) A neutrosophic set $V = \{(x, f_V(x), g_V(x), h_V(x))\}$ in (W, ρ) is said to be neutrosophic regular closed (NRCS, for short) when $V = N_{cl}(N_{int}(V))$ and its complement is called neutrosophic regular open set.

3 Neutrosophic Weak Compactness via Grills

Here we would investigate neutrosophic weak compactness via grills.

Definition 11 A neutrosophic open cover $\{O_i : i \in J\}$ of $(W, \mathscr{G}, \rho)$ is known as neutrosophic regular when for every $i \in J$, there exists a non-empty neutrosophic regular closed set F_i in W for which $F_i \subset O_i$ and $W = \bigcup\{\mathrm{int}(F_i) : i \in J\}$.

Definition 12 Consider $(W, \rho, \mathscr{G})$ as a neutrosophic topological space with grill $\mathscr{G}$. Then it is called neutrosophic $\mathscr{G}$-weakly compact if each neutrosophic regular cover of W has subfamily $\{V_1, V_2, \ldots, V_n\}$ where $W \setminus \bigcup_{i=1}^{n} \mathrm{cl}(V_i) \notin \mathscr{G}$.

Throughout the article $(W, \rho, \mathscr{G})$ denotes neutrosophic topological space via grill unless otherwise stated.

Definition 13 A neutrosophic set V of $(W, \rho, \mathscr{G})$ is called neutrosophic $\mathscr{G}$-weakly compact if V is a neutrosophic $\mathscr{G}$-weakly compact subspace of W.

Definition 14 A subset V of $(W, \rho, \mathscr{G})$ is called neutrosophic $\mathscr{G}$-weakly compact relative to W if for each neutrosophic regular cover $\{O_i : i \in J\}$ of V by neutrosophic open sets of W, there exists a finite subset J_0 of J such that $V \setminus \bigcup\{\mathrm{cl}(O_i) : i \in J_0\} \notin \mathscr{G}$.

Definition 15 The space $(W, \rho, \mathscr{G})$ is called neutrosophic $\mathscr{G}$-almost compact when every neutrosophic cover of W has subfamily $\{V_1, V_2, \ldots, V_n\}$ such that $W \setminus \bigcup_{i=1}^{n} \mathrm{cl}(V_i) \notin \mathscr{G}$.

Remark 2 Each neutrosophic $\mathscr{G}$-almost compact space is neutrosophic $\mathscr{G}$-weakly compact.

Theorem 1 *Consider $(W, \rho, \mathscr{G})$ where $\mathscr{G}$ is a σ-grill. If V is neutrosophic open and $\mathscr{G}$-weakly compact space of $(W, \rho, \mathscr{G})$, then V is neutrosophic $\mathscr{G}$-weakly compact relative to W.*

Proof Consider $\{O_i : i \in J\}$ as a neutrosophic regular cover of V by neutrosophic open sets of (W, ρ). So there exists a non-empty neutrosophic regular closed set F_i such that $F_i \subset O_i$ and $V \subset \bigcup\{\text{int}(F_i) : i \in J\}$. For each $i \in J$, $\text{int}(F_i) \cap V$ is a neutrosophic θ-open cover. Also $F_i \cap F$ is closed in V. But $\text{cl}_V(\text{int}(F_i) \cap V) \subset F_i \cap V \subset O_i$. Moreover, $V = \bigcup\{\text{int}(F_i) \cap V\}$ and $\text{int}(F_i) \cap V \subset \text{int}_V(\text{cl}_V(\text{int}(F_i)) \cap V)$. But $\text{cl}_V(\text{int}(F_i) \cap V)$ is neutrosophic regular closed in V and $\{O_i \cap V : i \in J\}$ is a neutrosophic regular cover of the subspace V. Since V is neutrosophic $\mathcal{G}$-weakly compact, there exists a finite J_0 of J where $V \setminus \bigcup_{i \in J_0}\{\text{cl}_V(O_i \cap V)\} \notin \mathcal{G}$. Also $\text{cl}_V(O_i \cap V) \subset \text{cl}(O_i)$ for each $i \in J_0$. Thus a finite collection is obtained where $V \setminus \bigcup_{i \in J_0}\{\text{cl}(O_i)\} \notin \mathcal{G}$. □

Theorem 2 *Consider $(W, \rho, \mathcal{G})$ where $\mathcal{G}$ is a σ-grill. If each proper neutrosophic regular closed subset of it is neutrosophic $\mathcal{G}$-weakly compact relative to W, then W is neutrosophic $\mathcal{G}$-weakly compact.*

Proof Consider $O_i : i \in J$ as a neutrosophic regular cover of $(W, \rho, \mathcal{G})$. Then for every $i \in J$, there exists a non-empty neutrosophic regular closed set F_i in W such that $F_i \subset O_i$ and $W = \bigcup \text{int}(F_i) : i \in J$. Consider i_0 where $i_0 \in J$. Take $P = W \setminus \text{int}(F_{i_0})$. Thus P is a neutrosophic regular closed in W and $P \subset \bigcup \text{int}(F_i) : i \in J_0$. Therefore, $O_i : i \in J \setminus i_0$ is a neutrosophic cover of P by neutrosophic open sets of W obeying neutrosophic regular cover property. Thus for some finite subset J_0 of J, we obtain $P \setminus \bigcup \text{cl}(V_i) : i \in J_0 \notin \mathcal{G}$. But we know that $W = P \cup (W - P)$. Also $W = P \cup \text{int}(F_{i_0})$. That is $W = P \cup O_{i_0}$. Thus we obtain $W \setminus \bigcup \text{int}(O_i) : i \in J_0 \cup i_0 \notin \mathcal{G}$. So W is neutrosophic $\mathcal{G}$-weakly compact. □

Corollary 1 *When each proper neutrosophic regular closed subset of $(W, \rho, \mathcal{G})$ is neutrosophic $\mathcal{G}$-weakly compact, so is W.*

Theorem 3 *Consider $(W, \rho, \mathcal{G})$ where $\mathcal{G}$ is a σ-grill and it is neutrosophic $\mathcal{G}$-weakly compact. A neutrosophic closed set V of W is neutrosophic $\mathcal{G}$-weakly compact relative to W.*

Proof Consider $\{O_i : i \in J\}$ as a neutrosophic regular cover of V. Take $\{W - V\} \neq 0_\eta$. As V is neutrosophic closed in W, $W - V$ is neutrosophic open in W. Thus, the collection $\{O_i : i \in J\} \cup \{W - V\}$ is a neutrosophic regular cover of W. As W is neutrosophic $\mathcal{G}$-weakly compact, there exists a finite subset J_0 of J where $W \setminus \left[\bigcup_{i \in J_0} \text{cl}(O_i) \cup \text{cl}(W - V)\right] \notin \mathcal{G}$. But we know that $V \nsubseteq (W - V)$. Then $V \setminus \left[\bigcup_{i \in J_0} \text{cl}(O_i)\right] \notin \mathcal{G}$. Hence the proof. □

Definition 16 Let $(W_1, \rho_1, \mathcal{G}_1)$ and $(W_2, \rho_2, \mathcal{G}_2)$ be two neutrosophic topological spaces with grills $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. A function $f : (W_1, \rho_1, \mathcal{G}_1) \to (W_2, \rho_2, \mathcal{G}_2)$ is said to be neutrosophic almost open if and only if $f^{-1}(\text{cl}(V)) \subset \text{cl}(f^{-1}(V))$ for every neutrosophic open set V of W_2.

Theorem 4 *Let $f : (W_1, \rho_1, \mathcal{G}_1) \to (W_2, \rho_2, \mathcal{G}_2)$ be a neutrosophic almost open perfect function. If P is neutrosophic $\mathcal{G}_2$-weakly compact relative to W_2, then $f^{-1}(P)$ is neutrosophic $\mathcal{G}_1$-weakly compact relative to W_1, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are σ-grills.*

Proof Consider P as neutrosophic $\mathscr{G}_2$ weakly compact relative to W_2 and $O_i : i \in J$ is any neutrosophic open regular cover of $f^{-1}(P)$. For each $i \in J$, there exists a neutrosophic regular closed set F_i of W_1 such that $F_i \subset O_i$ and $f^{-1}(P) \subset \bigcup \text{int}(F_i) : i \in J$. As f is a perfect function, for every $p \in P$, $f^{-1}(p)$ is neutrosophic compact. There exists a neutrosophic finite subset $J(p)$ of J such that $f^{-1}(p) \subset \bigcup \text{int}(F_i) : i \in J(p)$. Therefore, $f^{-1}(p) \subset \bigcup_{i \in J(p)} \text{int}(F_i) \subset \bigcup_{i \in J(p)} F_i \subset \bigcup_{i \in J(p)} O_i$. Define $G_p = W_2 - f(W_1 - (\text{int}(F_i) : i \in J(p)))$ $H_p = W_2 - \text{intcl}(f(W_1 - \bigcup \text{int}(F_i) : i \in J(p))$ $V_p = W_2 - f W_1 - \bigcup O_i : i \in J(p)$. The closeness of f implies that V_p are neutrosophic open in W_2. Since f is neutrosophic almost open, H_p is neutrosophic regular closed in W_2 and $p \in G_p \subset H_p \subset V_p$. The family $V_p : p \in P$ is a neutrosophic regular cover of neutrosophic open sets. Since P is neutrosophic $\mathscr{G}_2$ weakly compact relatively to W_2, there exists a finite number of neutrosophic points of P, say $p_1, p_2, ..., p_n$, such that $P \setminus \bigcup \text{cl}(V p_i) : i = 1, 2, ..., n \notin \mathscr{G}_2$. Therefore, $f^{-1}(P) \setminus \bigcup \text{cl}(O_i) : i \in J(p_i), i = 1, 2, ..., n \notin \mathscr{G}_1$. This shows that $f^{-1}(P)$ is neutrosophic $\mathscr{G}_1$ weakly compact relatively to W_1. □

4 Neutrosophic Weak Compactness by Neutrosophic θ-Open Sets

Here we would study neutrosophic weak compactness with the help of neutrosophic, θ-open sets

Definition 17 A neutrosophic space (W, ρ) is called neutrosophic almost regular when for every neutrosophic point $u \in W$ and every neutrosophic neighbourhood V of u, there exists a neutrosophic neighbourhood U of another neutrosophic point p of W such that $U \subset intcl(V)$.

Definition 18 A neutrosophic θ-open cover $\{O_i : i \in J\}$ of (W, ρ) is said to be neutrosophic θ-regular, if for every $i \in J$, there exists a non-empty neutrosophic regular closed set H_i in W such that $H_i \subset O_i$ and $W = \bigcup \{int(H_i) : i \in J\}$.

Definition 19 The space $(W, \rho, \mathscr{G})$ is called neutrosophic $\mathscr{G}$-θ weakly compact when each neutrosophic θ-regular cover of W has a subfamily $\{V_1, V_2, \ldots, V_n\}$ where $W \setminus \bigcup_{i=1}^{n} \text{cl}(V_i) \notin \mathscr{G}$.

Definition 20 A set S of $(W, \rho, \mathscr{G})$ is called neutrosophic $\mathscr{G}$-θ weakly compact when it is neutrosophic $\mathscr{G}$-θ weakly compact as the neutrosophic subspace of W.

Definition 21 A set A of $(W, \rho, \mathscr{G})$ is called neutrosophic $\mathscr{G}$-θ weakly compact with respect to W when every neutrosophic θ-regular cover $\{O_i : i \in J\}$ of A by neutrosophic θ-open sets of W has a finite subset J_0 of J where $A \setminus \bigcup \{\text{cl}(O_i) : i \in J_0\} \notin \mathscr{G}$.

Definition 22 The space $(W, \rho, \mathcal{G})$ is called neutrosophic $\mathcal{G}$-θ almost compact when each neutrosophic θ-open cover of W has a subfamily $\{V_1, V_2, \ldots, V_n\}$ such that $W \setminus \bigcup_{i=1}^{n} \mathrm{cl}(V_i) \notin \mathcal{G}$.

Remark 3 Each neutrosophic $\mathcal{G}$-θ almost compact space is neutrosophic $\mathcal{G}$-θ weakly compact.

Theorem 5 *If V is a neutrosophic θ-open set and neutrosophic $\mathcal{G}$-θ weakly compact subspace of $(W, \rho, \mathcal{G})$, then V is neutrosophic $\mathcal{G}$-θ weakly compact relative to W where $\mathcal{G}$ is a σ-grill.*

Proof Consider $O_i : i \in J$ as a neutrosophic θ-regular cover of V by neutrosophic θ-open sets of W. Hence there exists a non-empty neutrosophic regular closed set H_i such that $H_i \subset O_i$ and $V \subset \bigcup \mathrm{int}(H_i) : i \in J$. For each $i \in J$, $\mathrm{int}(H_i) \cap V$ is a neutrosophic θ-open cover. Again $H_i \cap V$ is neutrosophic closed in V. But $\mathrm{cl}_V(\mathrm{int}(H_i) \cap V) \subset H_i \cap V \subset O_i \cap V$. Further, $V = \bigcup \mathrm{int}(H_i) \cap V$ and $\mathrm{int}(H_i) \cap V \subset \mathrm{int}_V(\mathrm{cl}_V(\mathrm{int}(H_i) \cap V))$. But $\mathrm{cl}_V(\mathrm{int}(H_i) \cap V)$ is neutrosophic regular closed in V and $O_i \cap V : i \in J$ is a neutrosophic θ-regular cover of the subspace V. Since V is neutrosophic $\mathcal{G}$-θ weakly compact as a subspace, there exists a finite subset J_0 of J such that $V \setminus \bigcup_{i \in J_0} \mathrm{cl}_V(O_i \cap V) \notin \mathcal{G}$. Also $\mathrm{cl}_V(O_i \cap V) \subset \mathrm{cl}(O_i)$ for each $i \in J_0$. Thus we can obtain a finite subcollection where $V \setminus \bigcup_{i \in J_0} \mathrm{cl}_V(O_i) \notin \mathcal{G}$. □

Theorem 6 *If each proper neutrosophic regular closed set of $(W, \rho, \mathcal{G})$ is neutrosophic $\mathcal{G}$-θ weakly compact relative to W, then W is neutrosophic $\mathcal{G}$-θ weakly compact, where $\mathcal{G}$ is a σ-grill.*

Proof Consider $O_i : i \in J$ as a neutrosophic θ-regular cover of W. Hence for each $i \in J$, there exists a non-empty neutrosophic regular closed set H_i in W where $H_i \subset O_i$ and $W = \bigcup \mathrm{int}(H_i) : i \in J$. Choose i_0 where $i_0 \in J$. Take $P = W - \mathrm{int}(H_{i_0})$. Then P is a neutrosophic regular closed set in W and $P \subset \bigcup \mathrm{int}(H_i) : i \in J - i_0$. Therefore, $V_i : i \in J - i_0$ is a cover of P by neutrosophic θ-open sets on W obeying neutrosophic θ-regular cover property. For some finite subset J_0 of J, we have $P \setminus \bigcup \mathrm{cl}(O_i) : i \in J_0 \notin \mathcal{G}$. Again, $W = P \cup (W - P)$. Further, $W = P \cup \mathrm{int}(H_{i_0})$. So $W = P \cup V_{i_0}$. Thus, $W \setminus \bigcup \mathrm{cl}(O_i) : i \in J_0 \cup i_0 \notin \mathcal{G}$. Hence W is neutrosophic $\mathcal{G}$-θ weakly compact. □

Corollary 2 *If every proper neutrosophic regular closed set of $(W, \rho, \mathcal{G})$ is neutrosophic $\mathcal{G}$-θ weakly compact, then W is neutrosophic $\mathcal{G}$-θ weakly compact.*

Theorem 7 *Let $(W, \rho, \mathcal{G})$ be neutrosophic $\mathcal{G}$-θ weakly compact. If V is a neutrosophic θ-closed set of W, then it is neutrosophic $\mathcal{G}$-θ weakly compact relative to W where $\mathcal{G}$ is a σ-grill.*

Proof Let $\{O_i : i \in J\}$ be a neutrosophic θ-regular cover of V by neutrosophic θ-open sets. Consider $W - V \neq 0_\eta$. As V is neutrosophic θ-closed in W, $W - V$ is a neutrosophic θ-open set in W. So $\{O_i : i \in J\} \cup \{W - V\}$ is a neutrosophic θ-regular cover of W by neutrosophic θ-open sets. As W is neutrosophic $\mathcal{G}$-θ weakly compact, there exists a finite subset J_0 of J where $W \setminus \left[\bigcup_{i \in J_0} \mathrm{cl}(O_i) \cup \mathrm{cl}(W - V)\right] \notin \mathcal{G}$. As $V \nsubseteq (W - V)$, so $V \setminus \left[\bigcup_{i \in J_0} \mathrm{cl}(O_i)\right] \notin \mathcal{G}$. Thus V is neutrosophic $\mathcal{G}$-θ weakly compact. □

Definition 23 Let $(W_1, \rho_1, \mathscr{G}_1)$ and $(W_2, \rho_2, \mathscr{G}_2)$ be two neutrosophic topological spaces with grills $\mathscr{G}_1$ and $\mathscr{G}_2$ respectively. A function $f : (W_1, \rho_1, \mathscr{G}_1) \to (W_2, \rho_2, \mathscr{G}_2)$ is called neutrosophic almost θ-open if and only if $f^{-1}(\text{cl}(F)) \subset \text{cl}(f^{-1}(F))$ for every neutrosophic θ-open set F of W_2.

Theorem 8 *Let $(W_1, \rho_1, \mathscr{G}_1)$ and $(W_2, \rho_2, \mathscr{G}_2)$ be two neutrosophic topological spaces with grills $\mathscr{G}_1$ and $\mathscr{G}_2$ respectively. Let $f : (W_1, \rho_1, \mathscr{G}_1) \to (W_2, \rho_2, \mathscr{G}_2)$ be a neutrosophic almost θ-open perfect function. If P is neutrosophic $\mathscr{G}_2$-weakly compact relative to W_2, then $f^{-1}(P)$ is neutrosophic $\mathscr{G}_1$-weakly compact relative to W_1 where $\mathscr{G}_1$ and $\mathscr{G}_2$ are σ-grills.*

Proof Consider P as a neutrosophic $\mathscr{G}_2$-weakly compact relative to W_2 and $O_i : i \in J$ a neutrosophic θ-open regular cover of $f^{-1}(P)$. For each $i \in J$, there exists a neutrosophic regular closed set H_i of W_1 such that $H_i \subset O_i$ and $f^{-1}(P) \subset \bigcup \text{int}(H_i) : i \in J$. As f is perfect, for every $p \in P$, $f^{-1}(p)$ is neutrosophic compact. There exists a finite subset $J(p)$ of J where $f^{-1}(p) \subset \bigcup \text{int}(H_i) : i \in J(p)$. Thus, $f^{-1}(p) \subset \bigcup_{i \in J(p)} \text{int}(H_i) \subset \bigcup_{i \in J(p)} H_i \subset \bigcup_{i \in J(p)} O_i$. Define $G_p = W_2 - f(W_1 - \text{int}(H_i) : i \in J(p))$ $H_p = W_2 - \text{int}(\text{cl}(f(W_1 - \bigcup H_i : i \in J(p))))$ $V_p = W_2 - f(W_1 - (\bigcup O_i : i \in J(p)))$. The closeness of f gives V_p are neutrosophic θ-open in W_2. Since f is almost neutrosophic θ-open, H_p is neutrosophic regular closed in W_2 and $p \in G_p \subset H_p \subset V_p$. The set $V_p : p \in P$ is a neutrosophic regular cover by neutrosophic θ-open sets. As P is neutrosophic $\mathscr{G}_2$-θ-weakly compact relative to W_2, there exists a finite number of neutrosophic points of P, say $p_1, p_2, ..., p_n$, such that $P \setminus \bigcup \text{cl}(V p_i) : i = 1, 2, ..., n \notin \mathscr{G}_2$. Thus, $f^{-1}(P) \setminus \bigcup \text{cl}(O_i) : i \in J(p_i), i = 1, 2, ..., n \notin \mathscr{G}_1$. This indicates $f^{-1}(P)$ is neutrosophic $\mathscr{G}_1$-θ-weakly compact relative to W_1. □

5 Conclusion

We have studied the notion of compactness via grills in a neutrosophic topological space. We have defined neutrosophic weak $\mathscr{G}$-compactness and investigated some basic properties, theorems and results of this newly defined compactness in a neutrosophic topological space. We have obtained its relationship with other known neutrosophic compactness. We have exhibited neutrosophic weak compactness through neutrosophic θ-open sets. We hope this paper will motivate for further research in science and engineering.

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On $e^*\theta$-Regular and $e^*\theta$-Normal Spaces

Burcu Sünbül Ayhan

Abstract The purpose of this study is to introduce a new class of regular spaces called $e^*\theta$-regular spaces which is a generalization of the class of $\beta\theta$-regular spaces [5]. Also, we investigate some basic properties and several characterizations of $e^*\theta$-regular and $e^*\theta$-normal [3] spaces. Moreover, some functions such as $e^*\theta$-closed function and generalized $e^*\theta$-closed function, generalized $e^*\theta$-closed function have been defined and studied. Furthermore, we obtain some preservation theorems.

1 Introduction

Functional analysis is the starting point for many different fields of mathematics. One of them abstracted the concept of open ball to open set and thus a topology. Hence, many of the results were generalized to a much larger setting.

Regular and normal space concepts, which have an important role in the classification of topological spaces, were introduced and studied respectively by Vietoris [25] and Tietze [23]. $e^*\theta$-regular spaces which are a generalization of regular spaces and $e^*\theta$-normal spaces which have been defined by Ayhan and Özkoç [3] as a generalization of regular spaces have been defined via e^*-θ-open sets [10]. In recent years, many authors have studied several forms of regularity; such as $\beta\theta$-regularity [5], semiregularity [13], preregularity [9], b-regularity [20], e-regularity [18], β-regularity [17] and normality; such as $\beta\theta$-normality [5], seminormality [14], prenormality [21], e-normality [8], e^*-normality [8] and $e^*\theta$-normality [3].

The main goal of this book chapter is to study the notions of $e^*\theta$-regular and $e^*\theta$-normal spaces and obtain some of their characterizations.

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2 $e^*\theta$-Regular and $e^*\theta$-Normal Spaces

Regular and normal spaces are two of the most important topological properties in topological spaces. In this chapter, the concepts of $e^*\theta$-regular and $e^*\theta$-normal [3] spaces will be discussed.

2.1 Preliminaries

Throughout the manuscript, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open) sets of X is denoted by $C(X)$ (resp. $O(X)$). A subset A is said to be regular open [22] (resp. regular closed [22]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point $x \in X$ is said to be δ-cluster point [24] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighborhood U of x. The set of all δ-cluster points of A is called the δ-closure [24] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ-closed [24] and the complement of a δ-closed set is called δ-open [24]. The set $\{x|(\exists U \in O(X, x))(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by $int_\delta(A)$.

A subset A is called semiopen [12] (resp. preopen [15], b-open [2], β-open [1], e-open [6], e^*-open [7]) if $A \subseteq cl(int(A))$ (resp. $A \subseteq int(cl(A))$, $A \subseteq cl(int(A)) \cup int(cl(A))$, $A \subseteq cl(int(cl(A)))$, $A \subseteq cl(int_\delta(A)) \cup int(cl_\delta(A))$, $A \subseteq cl(int(cl_\delta(A)))$). The complement of a semiopen (resp. preopen, b-open, β-open, e-open, e^*-open) set is called semiclosed [12] (resp. preclosed [15], b-closed [2], β-closed [1], e-closed [6], e^*-closed [7]). The intersection of all semiclosed (resp. preclosed, b-closed, β-closed, e-closed, e^*-closed) sets of X containing A is called the semi-closure [12] (resp. pre-closure [15], b-closure [2], β-closure [1], e-closure [6], e^*-closure [7]) of A and is denoted by $scl(A)$ (resp. $pcl(A)$, $bcl(A)$, $\beta cl(A)$, $e\text{-}cl(A)$, $e^*\text{-}cl(A)$). The union of all semiopen (resp. preopen, b-open, β-open, e-open, e^*-open) sets of X contained in A is called the semi-interior [12] (resp. pre-interior [15], b-interior [2], β-interior [1], e-interior [6], e^*-interior [7]) of A and is denoted by $sint(A)$ (resp. $pint(A)$, $bint(A)$, $\beta int(A)$, $e\text{-}int(A)$, $e^*\text{-}int(A)$).

A point x of X is called an e^*-θ-cluster (β-θ-cluster) point of A if $e^*\text{-}cl(U) \cap A \neq \emptyset$ ($\beta cl(U) \cap A \neq \emptyset$) for every e^*-open (β-open) set U containing x. The set of all e^*-θ-cluster (β-θ-cluster) points of A is called the e^*-θ-closure [10] (β-θ-closure [17]) of A and is denoted by $e^*\text{-}cl_\theta(A)$ ($\beta cl_\theta(A)$). A subset A is said to be e^*-θ-closed (β-θ-closed) if $A = e^*\text{-}cl_\theta(A)$ ($A = \beta cl_\theta(A)$). The complement of an e^*-θ-closed (β-θ-closed) set is called e^*-θ-open [10] (β-θ-open [17]). A point x of X said to be an e^*-θ-interior [10] (β-θ-interior [17]) point of a subset A, denoted by $e^*\text{-}int_\theta(A)$ ($\beta int_\theta(A)$), if there exists an e^*-open (β-open) set U of X containing x such that $e^*\text{-}cl(U) \subseteq A$ ($\beta cl(U) \subseteq A$). A subset A is said to be e^*-θ-regular [10] (resp. β-θ-regular [17], e^*-regular [10]) set if it is e^*-θ-open (resp. β-θ-open, e^*-open) and e^*-θ-closed (resp. β-θ-closed, e^*-closed).

Also it is noted in [10] that

$$e^*\text{-regular} \Rightarrow e^*\text{-}\theta\text{-open} \Rightarrow e^*\text{-open}.$$

The family of all e^*-θ-open (resp. e^*-θ-closed, e^*-θ-regular, β-θ-open, β-θ-closed, e^*-open, e^*-closed, e^*-regular, regular open, regular closed, δ-open, δ-closed, semi open, semiclosed, preopen, preclosed, b-open, b-closed, β-open, β-closed, e-open, e-closed) subsets of X is denoted by $e^*\theta O(X)$ (resp. $e^*\theta C(X)$, $e^*\theta R(X)$, $\beta\theta O(X)$, $\beta\theta C(X)$, $e^*O(X)$, $e^*C(X)$, $e^*R(X)$, $RO(X)$, $RC(X)$, $\delta O(X)$, $\delta C(X)$, $SO(X)$, $SC(X)$, $PO(X)$, $PC(X)$, $BO(X)$, $BC(X)$, $\beta O(X)$, $\beta C(X)$, $eO(X)$, $eC(X)$). The family of all open (resp. closed, e^*-θ-open, e^*-θ-closed, e^*-θ-regular, β-θ-open, β-θ-closed, e^*-open, e^*-closed, e^*-regular, regular open, regular closed, δ-open, δ-closed, semiopen, semiclosed, preopen, preclosed, b-open, b-closed, β-open, β-closed, e-open, e-closed) sets of X containing a point x of X is denoted by $O(X, x)$ (resp. $C(X, x)$, $e^*\theta O(X, x)$, $e^*\theta C(X, x)$, $e^*\theta R(X, x)$, $\beta\theta O(X, x)$, $\beta\theta C(X, x)$, $e^*O(X, x)$, $e^*C(X, x)$, $e^*R(X, x)$, $RO(X, x)$, $RC(X, x)$, $\delta O(X, x)$, $\delta C(X, x)$, $SO(X, x)$, $SC(X, x)$, $PO(X, x)$, $PC(X, x)$, $BO(X, x)$, $BC(X, x)$, $\beta O(X, x)$, $\beta C(X, x)$, $eO(X, x)$, $eC(X, x)$).

We shall use the well-known accepted language almost in the whole of the proofs of the theorems in this book chapter. The following basic properties of e^*-θ-closure are useful in the sequel.

Lemma 1 ([10, 11]) *Let A be a subset of a space X, then the following properties hold:*

(1) $A \subseteq e^*\text{-}cl(A) \subseteq e^*\text{-}cl_\theta(A)$;
(2) *If* $A \in e^*O(X)$, *then* $e^*\text{-}cl_\theta(A) = e^*\text{-}cl(A)$;
(3) *If* $A \subseteq B$, *then* $e^*\text{-}cl_\theta(A) \subseteq e^*\text{-}cl_\theta(B)$;
(4) $e^*\text{-}cl_\theta(A) \in e^*\theta C(X)$ *and* $e^*\text{-}cl_\theta(e^*\text{-}cl_\theta(A)) = e^*\text{-}cl_\theta(A)$;
(5) *If* $A \in e^*\theta C(X)$, *then* $e^*\text{-}cl_\theta(A) = A$;
(6) *If* $A \in e^*\theta C(X)$, *then* $\backslash A \in e^*\theta O(X)$;
(7) *If* $A_\alpha \in e^*\theta C(X)$ *for each* $\alpha \in \Lambda$, *then* $\bigcap\{A_\alpha | \alpha \in \Lambda\} \in e^*\theta C(X)$;
(8) *If* $A_\alpha \in e^*\theta O(X)$ *for each* $\alpha \in \Lambda$, *then* $\bigcup\{A_\alpha | \alpha \in \Lambda\} \in e^*\theta O(X)$;
(9) $e^*\text{-}cl_\theta(A) = \bigcap\{U | (A \subseteq U)(U \in e^*\theta C(X))\}$;
(10) $e^*\text{-}cl_\theta(\backslash A) = \backslash e^*\text{-}int_\theta(A)$ *and* $e^*\text{-}int_\theta(\backslash A) = \backslash e^*\text{-}cl_\theta(A)$.

Lemma 2 ([16]) *For a topological space X, the following statements are equivalent:*

(1) *X is regular;*
(2) $(\forall x \in X)[U \in O(X, x) \Rightarrow (\exists V \in O(X, x))(cl(V) \subseteq U)]$.

Definition 1 A topological space is said to be

(a) p-regular [9] if for each closed set F of X and each point $x \notin F$, there exist disjoint preopen sets U and V such that $F \subseteq U$ and $x \in V$.
(b) s-regular [13] if for each closed set F of X and each point $x \notin F$, there exist disjoint semiopen sets U and V such that $F \subseteq U$ and $x \in V$.

(c) b-regular [20] if for each closed set F of X and each point $x \notin F$, there exist disjoint b-open sets U and V such that $F \subseteq U$ and $x \in V$.
(d) β-regular [17] if for each closed set F of X and each point $x \notin F$, there exist disjoint β-open sets U and V such that $F \subseteq U$ and $x \in V$.
(e) $\beta\theta$-regular [5] if for each closed set F of X and each point $x \notin F$, there exist disjoint β-θ-open sets U and V such that $F \subseteq U$ and $x \in V$.
(f) e-regular [18] if for each closed set F of X and each point $x \notin F$, there exist disjoint e-open sets U and V such that $F \subseteq U$ and $x \in V$.
(g) e^*-regular if for each closed set F of X and each point $x \notin F$, there exist disjoint e^*-open sets U and V such that $F \subseteq U$ and $x \in V$.

2.2 $e^*\theta$-Regular and (e^*, θ)-Regular Spaces

In this section, two new classes of regular spaces called $e^*\theta$-regular and (e^*, θ)-regular spaces are introduced and studied through the concept of e^*-θ-open set defined by Farhan and Yang [10]. In addition, not only some of their characterizations, but also their relations with other existing generalized regular space types are investigated.

Definition 2 A topological space is said to be $e^*\theta$-regular if for each closed set F of X and each point $x \notin F$, there exist disjoint e^*-θ-open sets U and V such that $F \subseteq U$ and $x \in V$.

Remark 1 From Definitions 1 and 2, we have the following diagram. However, none of the below implications is reversible as shown in the relevant articles and the following example:

$$\begin{array}{ccccccc} p\text{-regular} & \longrightarrow & e\text{-regular} & \longrightarrow & e^*\text{-regular} & \longleftarrow & e^*\theta\text{-regular} \\ & \searrow & & & \uparrow & & \uparrow \\ s\text{-regular} & \longrightarrow & b\text{-regular} & \longrightarrow & \beta\text{-regular} & \longleftarrow & \beta\theta\text{-regular} \end{array}$$

Example 1 Let $X = \{1, 2, 3, 4\}$ and $\tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3, 4\}\}$. It is not difficult to see $e^*\theta O(X) = 2^X$ and $\beta\theta O(X) = \{\emptyset, X, \{2\}, \{1, 3, 4\}\}$. Then, X is $e^*\theta$-regular but it is not $\beta\theta$-regular.

Question: Is there any e^*-regular topological space which is not $e^*\theta$-regular space?

Theorem 1 *For a topological space X, the following statements are equivalent:*

(1) X is $e^*\theta$-regular;
(2) For each point x in X and each open set U of X containing x, there exists $V \in e^*\theta O(X, x)$ such that $e^*\text{-}cl_\theta(V) \subseteq U$;
(3) For each closed set F of X, $F = \bigcap\{e^*\text{-}cl_\theta(V) | (F \subseteq V)(V \in e^*\theta O(X))\}$;
(4) For each subset A of X and each open set U of X such that $A \cap U \neq \emptyset$, there exists $V \in e^*\theta O(X)$ such that $A \cap V \neq \emptyset$ and $e^*\text{-}cl_\theta(V) \subseteq U$;

(5) For each nonempty subset A of X and each closed set F of X such that $A \cap F = \emptyset$, there exist $V, W \in e^*\theta O(X)$ such that $A \cap V \neq \emptyset$, $F \subseteq W$ and $V \cap W = \emptyset$.

Proof (1) $\Rightarrow$ (2) : Let $U \in O(X, x)$.

$$\left.\begin{array}{r} U \in O(X,x) \Rightarrow (\backslash U \in C(X))(x \in U) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists V \in e^*\theta O(X,x))(\exists W \in e^*\theta O(X))(\backslash U \subseteq W)(V \cap W = \emptyset)$

$\Rightarrow (\exists V \in e^*\theta O(X,x))(\exists W \in e^*\theta O(X))(\backslash U \subseteq W)(e^*\text{-}cl_\theta(V) \cap W = \emptyset)$

$\Rightarrow (\exists V \in e^*\theta O(X,x))(e^*\text{-}cl_\theta(V) \subseteq U)$.

(2) $\Rightarrow$ (3) : Let $F \in C(X)$ and $x \notin F$.

$F \in C(X) \Rightarrow F \subseteq \bigcap\{e^*\text{-}cl_\theta(V) | (F \subseteq V)(V \in e^*\theta O(X))\} \ldots (1)$

$$\left.\begin{array}{r} (F \in C(X))(x \notin F) \Rightarrow \backslash F \in O(X,x) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists U \in e^*\theta O(X,x))(U \subseteq e^*\text{-}cl_\theta(U) \subseteq \backslash F) \\ V := \backslash e^*\text{-}cl_\theta(U) \end{array}\right\} \overset{\text{Lemma 1}}{\Rightarrow}$$

$\Rightarrow (F \subseteq V)(V \in e^*\theta O(X))(x \notin e^*\text{-}cl_\theta(V))$

$\Rightarrow x \notin \bigcap\{e^*\text{-}cl_\theta(V) | (F \subseteq V)(V \in e^*\theta O(X))\}$

$\Rightarrow \bigcap\{e^*\text{-}cl_\theta(V) | (F \subseteq V)(V \in e^*\theta O(X))\} \subseteq F \ldots (2)$

(1), (2) $\Rightarrow F = \bigcap\{e^*\text{-}cl_\theta(V) | (F \subseteq V)(V \in e^*\theta O(X))\}$.

(3) $\Rightarrow$ (4) : Let $A \subseteq X$, $U \in O(X)$ and $A \cap U \neq \emptyset$.

$$\left.\begin{array}{r} U \in O(X) \Rightarrow \backslash U \in C(X) \\ A \cap U \neq \emptyset \Rightarrow (\exists x \in X)(x \in A \cap U) \Rightarrow x \in U \end{array}\right\} \overset{\text{Hypothesis}}{\Rightarrow}$$

$$\left.\begin{array}{r} \Rightarrow (\exists W \in e^*\theta O(X))(\backslash U \subseteq W)(x \notin e^*\text{-}cl_\theta(W)) \\ V := \backslash e^*\text{-}cl_\theta(W) \end{array}\right\} \overset{\text{Lemma 1}}{\Rightarrow}$$

$\Rightarrow (V \in e^*\theta O(X))(x \in V \cap A \neq \emptyset)(e^*\text{-}cl_\theta(V) \subseteq e^*\text{-}cl_\theta(\backslash W) = \backslash W \subseteq U)$.

(4) $\Rightarrow$ (5) : Let $F \in C(X)$, $A \neq \emptyset$ and $A \cap F = \emptyset$.

$$\left.\begin{array}{r} F \in C(X) \Rightarrow \backslash F \in O(X) \\ (A \neq \emptyset)(A \cap F = \emptyset) \Rightarrow A \cap (\backslash F) \neq \emptyset \end{array}\right\} \overset{\text{Hypothesis}}{\Rightarrow}$$

$$\left.\begin{array}{r} \Rightarrow (\exists V \in e^*\theta O(X))(A \cap V \neq \emptyset)(e^*\text{-}cl_\theta(V) \subseteq \backslash F) \\ W := \backslash e^*\text{-}cl_\theta(V) \end{array}\right\} \Rightarrow$$

$\Rightarrow (V, W \in e^*\theta O(X))(A \cap V \neq \emptyset)(F \subseteq W)(V \cap W = \emptyset)$.

(5) $\Rightarrow$ (1) : It can be easily proved. □

Definition 3 A topological space is said to be

(a) (e^*, θ)-regular if for each e^*-θ-regular set F of X and each point x in $X \backslash F$, there exist disjoint open sets U and V such that $F \subseteq U$ and $x \in V$.

(b) Extremally $e^*\theta$-disconnected (briefly, $ED^{e^*\theta}$) if $e^*\text{-}cl_\theta(U)$ is e^*-θ-open in X for every e^*-θ-open set U of X.

Remark 2 From Definitions 2 and 3, every regular and (e^*, θ)-regular spaces are $e^*\theta$-regular space. The converses of these implications are not true in general as shown by the following example.

Example 2 Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. It is not difficult to see $e^*\theta O(X) = e^*O(X) = 2^X \backslash \{\{c\}, \{d\}, \{c, d\}\}$. Then X is $e^*\theta$-regular but it is not regular and (e^*, θ)-regular.

Theorem 2 *For a topological space X, the following statements are equivalent:*

(1) X is (e^*, θ)-regular;
(2) For each x in X and any e^*-θ-regular set U of X containing x, there exists $V \in O(X, x)$ such that $cl(V) \subseteq U$.

Proof (1) $\Rightarrow$ (2) : Let $x \in X$ and $U \in e^*\theta R(X, x)$.

$$\left.\begin{array}{r}(x \in X)(U \in e^*\theta R(X, x)) \Rightarrow (\backslash U \in e^*\theta R(X))(x \in U) \\ X \text{ is } (e^*, \theta)\text{-regular}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists W \in O(X))(\exists V \in O(X, x))(\backslash U \subseteq W)(V \cap W = \emptyset)$
$\Rightarrow (\exists W \in O(X))(\exists V \in O(X, x))(\backslash U \subseteq W)(cl(V) \cap W = \emptyset)$
$\Rightarrow (\exists W \in O(X))(\exists V \in O(X, x))(cl(V) \subseteq \backslash W \subseteq U)$.

(2) $\Rightarrow$ (1) : Let $F \in e^*\theta R(X)$ and $x \notin F$.

$$\left.\begin{array}{r}x \notin F \in e^*\theta R(X) \Rightarrow U := \backslash F \in e^*\theta R(X, x) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists V \in O(X, x))(cl(V) \subseteq U)$

$$\left.\begin{array}{r}\Rightarrow (\exists V \in O(X, x))(F = \backslash U \subseteq \backslash cl(V)) \\ W := \backslash cl(V)\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists V \in O(X, x))(W \in O(X))(F \subseteq W)(V \cap W = \emptyset)$. □

Theorem 3 *If X is $e^*\theta$-regular, (e^*, θ)-regular and $ED^{e^*\theta}$, then it is regular.*

Proof Let $x \in O(X)$ and $x \in U$.

$$\left.\begin{array}{r}U \in O(X, x) \\ X \text{ is } e^*\theta\text{-regular}\end{array}\right\} \overset{\text{Theorem 1}}{\Rightarrow} \left.\begin{array}{r}(\exists V \in e^*\theta O(X, x))(e^*\text{-}cl_\theta(V) \subseteq U) \\ X \text{ is } ED^{e^*\theta}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow e^*\text{-}cl_\theta(V) \in e^*\theta R(X) \\ X \text{ is } (e^*, \theta)\text{-regular}\end{array}\right\} \overset{\text{Theorem 2}}{\Rightarrow}$$

$\Rightarrow (\exists W \in O(X, x))(cl(W) \subseteq e^*\text{-}cl_\theta(V) \subseteq U)$.
Then, X is regular from Lemma 2. □

Definition 4 A function $f : X \to Y$ is said to be

(a) $e^*\theta$-closed (resp. pre $e^*\theta$-closed [4]), if the image of each closed (resp. e^*-θ-closed) set F of X is e^*-θ-closed in Y.
(b) $e^*\theta$-open (resp. pre $e^*\theta$-open [4]), if the image of each open (resp. e^*-θ-open) set U of X is e^*-θ-open in Y.

Lemma 3 *A function $f : X \to Y$ is $e^*\theta$-closed (resp. pre $e^*\theta$-closed) if and only if for each subset B of Y and each open (resp. e^*-θ-open) set U containing $f^{-1}[B]$, there exists an e^*-θ-open set V of Y containing B such that $f^{-1}[V] \subseteq U$.*

Proof $(\Rightarrow)$: Let $B \subseteq Y$, $U \in O(X)$ and $f^{-1}[B] \subseteq U$.

$$\left.\begin{array}{r}(U \in O(X))(f^{-1}[B] \subseteq U) \Rightarrow (\backslash U \in C(X))(\backslash U \subseteq \backslash f^{-1}[B] = f^{-1}[\backslash B]) \\ f \text{ is } e^*\theta\text{-closed}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (f[\backslash U] \in e^*\theta C(Y))(f[\backslash U] \subseteq f[f^{-1}[\backslash B]] \subseteq \backslash B) \\ V := \backslash f[\backslash U]\end{array}\right\} \Rightarrow$$

$$\Rightarrow (V \in e^*\theta O(Y))(B \subseteq \backslash f[f^{-1}[\backslash B]] \subseteq V)(f^{-1}[V] \subseteq U).$$

$(\Leftarrow)$: Let $F \in C(X)$.

$$F \in C(X) \Rightarrow (f[F] \subseteq Y)(\backslash F \in O(X))(F \subseteq f^{-1}[f[F]])$$

$$\left.\begin{array}{r}\Rightarrow (\backslash f[F] \subseteq Y)(\backslash F \in O(X))(f^{-1}[\backslash f[F]] \subseteq \backslash F) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists V \in e^*\theta O(Y))(\backslash f[F] \subseteq V)(f^{-1}[V] \subseteq \backslash F)$$

$$\Rightarrow (\exists V \in e^*\theta O(Y))(\backslash V \subseteq f[F])(F \subseteq \backslash f^{-1}[V] = f^{-1}[\backslash V])$$

$$\Rightarrow (\exists V \in e^*\theta O(Y))(\backslash V \subseteq f[F])(f[F] \subseteq f[f^{-1}[\backslash V]] \subseteq \backslash V)$$

$$\Rightarrow (\backslash V \in e^*\theta C(Y))(\backslash V \subseteq f[F] \subseteq \backslash V)$$

$$\Rightarrow f[F] \in e^*\theta C(Y).$$

The other case can be proved similarly. □

Definition 5 A subset A of a space X is called generalized $e^*\theta$-closed (briefly, $ge^*\theta$-closed) if $e^*\text{-}cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X. A subset A of a space X is said to be generalized $e^*\theta$-open (briefly, $ge^*\theta$-open) if $X \backslash A$ is $ge^*\theta$-closed. The family of all $ge^*\theta$-closed (resp. $ge^*\theta$-open) subsets of X is denoted by $ge^*\theta C(X)$ (resp. $ge^*\theta O(X)$).

Lemma 4 *A subset A of a space X is $ge^*\theta$-open if and only if $F \subseteq e^*\text{-}int_\theta(A)$ whenever $F \subseteq A$ and F is closed in X.*

Proof $(\Rightarrow)$: Let $F \subseteq A$, $F \in C(X)$ and $A \in ge^*\theta O(X)$.

$$\left.\begin{array}{r}A \supseteq F \in C(X) \Rightarrow \backslash A \subseteq \backslash F \in O(X) \\ A \in ge^*\theta O(X) \Rightarrow \backslash A \in ge^*\theta C(X)\end{array}\right\} \Rightarrow$$

$$\Rightarrow \backslash e^*\text{-}int_\theta(A) = e^*\text{-}cl_\theta(\backslash A) \subseteq \backslash F$$

$$\Rightarrow F \subseteq e^*\text{-}int_\theta(A).$$

$(\Leftarrow)$: Let $\backslash F \in O(X)$ and $\backslash A \subseteq \backslash F$.

$$\left.\begin{array}{r}(\backslash F \in O(X))(\backslash A \subseteq \backslash F) \Rightarrow (F \in C(X))(F \subseteq A) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow F \subseteq e^*\text{-}int_\theta(A)$$

$$\Rightarrow e^*\text{-}cl_\theta(\backslash A) = \backslash e^*\text{-}int_\theta(A) \subseteq \backslash F.$$

Then, $\backslash A \in ge^*\theta C(X)$. Therefore, $A \in ge^*\theta O(X)$. □

Definition 6 A function $f : X \to Y$ is said to be

(a) generalized $e^*\theta$-closed (briefly, $ge^*\theta$-closed), if the image of each closed set F of X is $ge^*\theta$-closed in Y;
(b) generalized $e^*\theta$-open (briefly, $ge^*\theta$-open), if the image of each open set U of X is $ge^*\theta$-open in Y;
(c) pre-generalized $e^*\theta$-closed (briefly, pre $ge^*\theta$-closed), if the image of each $ge^*\theta$-closed set F of X is $ge^*\theta$-closed in Y;

(d) pre-generalized $e^*\theta$-open (briefly, pre $ge^*\theta$-open), if the image of each $ge^*\theta$-open set U of X is $ge^*\theta$-open in Y.

Lemma 5 *For a function $f : X \to Y$, the following statements are equivalent:*

(1) f is $ge^*\theta$-closed (resp. pre $ge^*\theta$-closed);
(2) For each subset B of Y and each open (resp. $ge^*\theta$-open) set U of X containing $f^{-1}[B]$, there exists a $ge^*\theta$-open set V of Y containing B such that $f^{-1}[V] \subseteq U$.

Proof It is similar to the proof of Lemma 3. □

Theorem 4 *If $f : X \to Y$ is a continuous $e^*\theta$-open and $ge^*\theta$-closed surjection from a regular space X onto a space Y, then Y is $e^*\theta$-regular.*

Proof Let $x \in X$, $y = f(x)$ and $U \in O(Y, y)$.

$$\left.\begin{array}{r}(y = f(x))(U \in O(Y, y)) \\ f \text{ is continuous}\end{array}\right\} \Rightarrow f^{-1}[U] \in O(X, x) \left.\begin{array}{r} \\ X \text{ is regular}\end{array}\right\} \overset{\text{Lemma 1}}{\Rightarrow}$$

$$\Rightarrow (\exists V \in O(X, x))(cl(V) \subseteq f^{-1}[U]) \left.\begin{array}{r} \\ f \text{ is an } e^*\theta\text{-open surjection}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (f[V] \in e^*\theta O(Y, y))(f[cl(V)] \subseteq f[f^{-1}[U]] = U) \left.\begin{array}{r} \\ f \text{ is} ge^*\theta\text{-closed}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (f[cl(V)] \in ge^*\theta C(Y, y))(e^*\text{-}cl_\theta(f[V]) \subseteq e^*\text{-}cl_\theta(f[cl(V)]) = f[cl(V)] \subseteq U).$$

Then, Y is $e^*\theta$-regular from Theorem 1 (2). □

2.3 $e^*\theta$-Normal and $(e^*, \theta)^*$-Normal Spaces

In this section, the fundamental properties of $e^*\theta$-normal spaces [3] and $(e^*, \theta)^*$-normal spaces are investigated. Additionally, several characterizations of both spaces have been obtained.

Definition 7 A topological space is said to be $e^*\theta$-normal [3] if for any pair of disjoint closed sets F_1 and F_2 of X, there exist disjoint e^*-θ-open sets U_1 and U_2 such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

Remark 3 From Definition 7, every normal space is an $e^*\theta$-normal space. The converse of this implication is not true in general as shown by the following example.

Example 3 Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, X\}$. It is not difficult to see that $e^*\theta O(X) = e^*O(X) = 2^X \setminus \{\{d\}\}$. Then, X is $e^*\theta$-normal but it is not normal.

Theorem 5 *For a topological space X, the following statements are equivalent:*

(1) X is $e^*\theta$-normal;
(2) For every pair of open sets U and V whose $U \cup V = X$, there exist $e^*\theta$-closed sets A and B such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$;
(3) For each closed set F and every open set G containing F, there exists $U \in e^*\theta O(X)$ such that $F \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq G$.

Proof (1) $\Rightarrow$ (2) : Let $U, V \in O(X)$ and $U \cup V = X$.

$$\left.\begin{array}{r}(U, V \in O(X))(U \cup V = X) \Rightarrow (\backslash U, \backslash V \in C(X))((\backslash U) \cap (\backslash V) = \emptyset) \\ X \text{ is } e^*\theta\text{-normal}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists U_1 \in e^*\theta O(X))(\exists V_1 \in e^*\theta O(X))(\backslash U \subseteq U_1)(\backslash V \subseteq V_1)(U_1 \cap V_1 = \emptyset) \\ (A := \backslash U_1)(B := \backslash V_1)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (A, B \in e^*\theta C(X))(A \subseteq U)(B \subseteq V)(A \cup B = X).$$

(2) $\Rightarrow$ (3) : Let $F \in C(X)$, $G \in O(X)$ and $F \subseteq G$.

$$\left.\begin{array}{r}(F \in C(X))(G \in O(X))(F \subseteq G) \Rightarrow (\backslash F \in O(X))(G \in O(X))((\backslash F) \cup G = X) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists W_1, W_2 \in e^*\theta C(X))(W_1 \subseteq \backslash F)(W_2 \subseteq G)(W_1 \cup W_2 = X)$$

$$\left.\begin{array}{r}\Rightarrow (\backslash W_1, \backslash W_2 \in e^*\theta O(X))(F \subseteq \backslash W_1)(\backslash G \subseteq \backslash W_2)((\backslash W_1) \cap (\backslash W_2) = \emptyset) \\ (U := \backslash W_1)(V := \backslash W_2)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (U \in e^*\theta O(X))(\backslash V \in e^*\theta C(X))(F \subseteq U \subseteq \backslash V \subseteq G)$$

$$\Rightarrow (U \in e^*\theta O(X))(F \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq e^*\text{-}cl_\theta(\backslash V) = \backslash V \subseteq G).$$

(3) $\Rightarrow$ (1) : Let $F_1, F_2 \in C(X)$ and $F_1 \cap F_2 = \emptyset$.

$$\left.\begin{array}{r}(F_1, F_2 \in C(X))(F_1 \cap F_2 = \emptyset) \Rightarrow (\backslash F_1, \backslash F_2 \in O(X))(F_1 \subseteq \backslash F_2) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists U \in e^*\theta O(X))(F_1 \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq \backslash F_2) \\ V := \backslash e^*\text{-}cl_\theta(U)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (U, V \in e^*\theta O(X))(F_1 \subseteq U)(F_2 \subseteq V)(U \cap V = \emptyset). \qquad \square$$

Theorem 6 *For a topological space X, the following statements are equivalent:*

(1) X is $e^*\theta$-normal;
(2) For every pair of open sets U and V whose $U \cup V = X$, there exist $ge^*\theta$-closed sets A and B such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$;
(3) For each closed set F and every open set G containing F, there exists $U \in ge^*\theta O(X)$ such that $F \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq G$;
(4) For each pair of disjoint closed sets F_1 and F_2 of X, there exist disjoint $ge^*\theta$-open sets U_1 and U_2 such that $F_1 \subseteq U_1$, $F_2 \subseteq U_2$.

Proof (1) $\Rightarrow$ (2) : The proof is clear from Theorem 5 since every e^*-θ-closed set is $ge^*\theta$-closed.

(2) $\Rightarrow$ (3) : Let $F \in C(X)$, $G \in O(X)$ and $F \subseteq G$.

$$\left.\begin{array}{r}(F \in C(X))(G \in O(X))(F \subseteq G) \Rightarrow (\backslash F, G \in O(X))((\backslash F) \cup G = X) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists A, B \in ge^*\theta C(X))(A \subseteq \backslash F)(B \subseteq G)(A \cup B = X)$$
$$\left.\begin{array}{r} \Rightarrow (\backslash A, \backslash B \in ge^*\theta O(X))(F \subseteq \backslash A)(\backslash G \subseteq \backslash B)((\backslash A) \cap (\backslash B) = \emptyset) \\ (U := e^*\text{-}int_\theta(\backslash A))(V := e^*\text{-}int_\theta(\backslash B)) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (U, V \in e^*\theta O(X))(F \subseteq U)(\backslash G \subseteq V)(U \cap V = \emptyset)$$
$$\Rightarrow (U \in ge^*\theta O(X))(F \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq \backslash V \subseteq G).$$

(3) $\Rightarrow$ (4) : Let $F_1, F_2 \in C(X)$ and $F_1 \cap F_2 = \emptyset$.

$$\left.\begin{array}{r} (F_1, F_2 \in C(X))(F_1 \cap F_2 = \emptyset) \Rightarrow (\backslash F_1, \backslash F_2 \in O(X))(F_1 \subseteq \backslash F_2) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (\exists U \in ge^*\theta O(X))(F_1 \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq \backslash F_2) \\ V := \backslash e^*\text{-}cl_\theta(U) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (U, V \in ge^*\theta O(X))(F_1 \subseteq U)(F_2 \subseteq V)(U \cap V = \emptyset).$$

(4) $\Rightarrow$ (1) : Let $F_1, F_2 \in C(X)$ and $F_1 \cap F_2 = \emptyset$.

$$\left.\begin{array}{r} (F_1, F_2 \in C(X))(F_1 \cap F_2 = \emptyset) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (\exists U_1, U_2 \in ge^*\theta O(X))(F_1 \subseteq U_1)(F_2 \subseteq U_2)(U_1 \cap U_2 = \emptyset) \\ \text{Lemma 1} \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (F_1 \subseteq e^*\text{-}int_\theta(U_1))(F_2 \subseteq e^*\text{-}int_\theta(U_2))(U_1 \cap U_2 = \emptyset) \\ (V_1 := e^*\text{-}int_\theta(U_1))(V_2 := e^*\text{-}int_\theta(U_2)) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (V_1, V_2 \in e^*\theta O(X))(F_1 \subseteq V_1)(F_2 \subseteq V_2)(V_1 \cap V_2 = \emptyset). \qquad \square$$

Definition 8 A topological space is said to be $e^*\theta$-R_0 if every open set of the space contains the e^*-θ-closure of each of its singletons.

Theorem 7 *If X is $e^*\theta$-normal and $e^*\theta$-R_0, then X is $e^*\theta$-regular.*

Proof Let $F \in C(X)$ and $x \notin F$.

$$\left.\begin{array}{r} x \notin F \in C(X) \Rightarrow \backslash F \in O(X, x) \\ X \text{ is } e^*\theta\text{-}R_0 \end{array}\right\} \Rightarrow \left.\begin{array}{r} K := e^*\text{-}cl_\theta(\{x\}) \subseteq \backslash F \\ X \text{ is } e^*\theta\text{-normal} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in e^*\theta O(X, x))(\exists V \in e^*\theta O(X))(K \subseteq U)(F \subseteq V)(U \cap V = \emptyset). \qquad \square$$

Definition 9 A topological space is said to be almost $e^*\theta$-irresolute if $f[e^*\text{-}cl_\theta(U)] = e^*\text{-}cl_\theta(f[U])$ for every $U \in e^*\theta O(X)$.

Theorem 8 *If $f : X \to Y$ is a pre $e^*\theta$-open continuous almost $e^*\theta$-irresolute surjection from an $e^*\theta$-normal space X onto a space Y, then Y is $e^*\theta$-normal.*

Proof Let $A \in C(Y)$, $B \in O(Y)$ and $A \subseteq B$.

$$\left.\begin{array}{r} (A \in C(Y))(B \in O(Y))(A \subseteq B) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (f^{-1}[A] \in C(X))(f^{-1}[B] \in O(X))(f^{-1}[A] \subseteq f^{-1}[B]) \\ X \text{ is } e^*\theta\text{-normal} \end{array}\right\} \overset{\text{Theorem 5(3)}}{\Rightarrow}$$
$$\left.\begin{array}{r} \Rightarrow (\exists U \in e^*\theta O(X))(f^{-1}[A] \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq f^{-1}[B]) \\ f \text{ is pre } e^*\theta\text{-open} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (f[U] \in e^*\theta O(Y))(f[f^{-1}[A]] \subseteq f[U] \subseteq f[e^*\text{-}cl_\theta(U)] \subseteq f[f^{-1}[B]]) \\ f \text{ is surjective} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (f[U] \in e^*\theta O(Y))(A \subseteq f[U] \subseteq f[e^*\text{-}cl_\theta(U)] \subseteq B) \\ f \text{ is almost } e^*\theta\text{-irresolute} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (f[U] \in e^*\theta O(Y))(A \subseteq f[U] \subseteq e^*\text{-}cl_\theta(f[U]) \subseteq B).$$

Then, Y is $e^*\theta$-normal from Theorem 5 (3). □

Theorem 9 *If $f : X \to Y$ is a $ge^*\theta$-closed (pre $ge^*\theta$-closed) continuous function from a normal ($e^*\theta$-normal) space X onto a space Y, then Y is $e^*\theta$-normal.*

Proof Let $A, B \in C(Y)$ and $A \cap B = \emptyset$.

$$\left.\begin{array}{r} (A, B \in C(Y))(A \cap B = \emptyset) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (f^{-1}[A], f^{-1}[B] \in C(X))(f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B] = \emptyset) \\ X \text{ is normal} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists U, V \in O(X))(f^{-1}[A] \subseteq U)(f^{-1}[B] \subseteq V)(U \cap V = \emptyset) \\ f \text{ is } ge^*\theta\text{-closed} \end{array}\right\} \overset{\text{Lemma 5}}{\Rightarrow}$$

$$\Rightarrow (\exists G, H \in ge^*\theta O(Y))(A \subseteq G)(B \subseteq H)(f^{-1}[G] \subseteq U)(f^{-1}[H] \subseteq V)(G \cap H = \emptyset).$$

Then, Y is $e^*\theta$-normal from Theorem 6 (4).

The other case can be proved similarly. □

Definition 10 A topological space X is said to be

(a) $e^*\theta$-T_0 [4] if for any distinct pair of points x and y in X, there is an e^*-θ-open set U in X containing x but not y or an e^*-θ-open set V in X containing y but not x.

(b) $e^*\theta$-T_1 [4] if for any distinct pair of points x and y in X, there is an e^*-θ-open set U in X containing x but not y and an e^*-θ-open set V in X containing y but not x.

(c) $e^*\theta$-T_2 [4] (resp. e^*-T_2 [8]) if for every pair of distinct points x and y, there exist two e^*-θ-open (resp. e^*-open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 10 ([4]) *For a topological space X, the following properties are equivalent:*

(1) X is $e^*\theta$-T_0;
(2) X is $e^*\theta$-T_1;
(3) X is $e^*\theta$-T_2;
(4) X is e^*-T_2;
(5) For every pair of distinct points $x, y \in X$, there exist $U \in e^*O(X, x)$ and $V \in e^*O(X, y)$ such that $e^*\text{-}cl(U) \cap e^*\text{-}cl(V) = \emptyset$;
(6) For every pair of distinct points $x, y \in X$, there exist $U \in e^*R(X, x)$ and $V \in e^*R(X, y)$ such that $U \cap V = \emptyset$;
(7) For every pair of distinct points $x, y \in X$, there exist $U \in e^*\theta O(X, x)$ and $V \in e^*\theta O(X, y)$ such that $e^*\text{-}cl_\theta(U) \cap e^*\text{-}cl_\theta(V) = \emptyset$.

Definition 11 A function $f : X \to Y$ is said to be

(a) $e^*\theta$-continuous [4] if $f^{-1}[V]$ is e^*-θ-closed in X for every closed set V in Y, equivalently if $f^{-1}[V]$ is e^*-θ-open in X for every open set V in Y.
(b) Strongly e^*-irresolute [19] if for each point $x \in X$ and each $V \in e^*O(Y, f(x))$, there exists $U \in e^*O(X, x)$ such that $f[e^*\text{-}cl(U)] \subseteq V)$.

Theorem 11 *A function $f : X \to Y$ is strongly e^*-irresolute if and only if $f^{-1}[V]$ is e^*-θ-open in X for every e^*-θ-open set V in Y.*

Proof It is obvious from Theorem 5.1(e) [19]. □

Theorem 12 *If $f : X \to Y$ is a strongly e^*-irresolute closed injection from a space X to an $e^*\theta$-normal space Y, then X is $e^*\theta$-normal.*

Proof Let $A, B \in C(X)$ and $A \cap B = \emptyset$.

$$\left.\begin{array}{r}(A, B \in C(X))(A \cap B = \emptyset)\\ f \text{ is closed injection}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (f[A], f[B] \in C(Y))(f[A] \cap f[B] = f[A \cap B] = \emptyset) \left.\begin{array}{r}\\ Y \text{ is } e^*\theta\text{-normal}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U, V \in e^*\theta O(Y))(f[A] \subseteq U)(f[B] \subseteq V)(U \cap V = \emptyset) \left.\begin{array}{r}\\ f \text{ is strongly } e^*\text{-irresolute}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (f^{-1}[U], f^{-1}[V] \in e^*\theta O(X))(A \subseteq f^{-1}[U])(B \subseteq f^{-1}[V])(f^{-1}[U] \cap f^{-1}[V] = \emptyset).$$ □

Theorem 13 *If $f : X \to Y$ is an $e^*\theta$-continuous closed injection from a space X to a normal space Y, then X is $e^*\theta$-normal.*

Proof It is similar to the proof of Theorem 12. □

Definition 12 A topological space is said to be $(e^*, \theta)^*$-normal if for any pair of disjoint e^*-θ-closed sets F_1 and F_2 of X, there exist disjoint e^*-θ-open sets U_1 and U_2 such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

Definition 13 A subset A of a space X is said to be (e^*, θ)-closed if $e^*\text{-}cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is e^*-θ-open in X. A subset A of a space X is said to be (e^*, θ)-open if $X \backslash A$ is (e^*, θ)-closed. The family of all (e^*, θ)-closed (resp. (e^*, θ)-open) subsets of X is denoted by $(e^*, \theta)C(X)$ (resp. $(e^*, \theta)O(X)$).

Lemma 6 *A subset A of a space X is (e^*, θ)-open if and only if $F \subseteq e^*\text{-}int_\theta(A)$ whenever $F \subseteq A$ and F is e^*-θ-closed in X.*

Proof It is similar to the proof of Lemma 4. □

Theorem 14 *For a topological space X, the following statements are equivalent:*

(1) X is $(e^*, \theta)^*$-normal;
(2) For every pair of e^*-θ-open sets U and V whose $U \cup V = X$, there exist e^*-θ-closed sets A and B such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$;
(3) For each e^*-θ-closed set F and every e^*-θ-open set G containing F, there exists an e^*-θ-open set U such that $F \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq G$;

(4) *For every pair of e^*-θ-open sets U and V whose $U \cup V = X$, there exist (e^*, θ)-closed sets A and B such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$;*
(5) *For each e^*-θ-closed set F and every e^*-θ-open set G containing F, there exists an (e^*, θ)-open set U such that $F \subseteq U \subseteq e^*\text{-}cl_\theta(U) \subseteq G$;*
(6) *For each pair of disjoint e^*-θ-closed sets F_1 and F_2 of X, there exist disjoint (e^*, θ)-open sets U_1 and U_2 such that $F_1 \subseteq U_1$, $F_2 \subseteq U_2$.*

Proof It is similar to the proof of Theorems 5 and 6. □

Definition 14 A function $f : X \to Y$ is said to be $(e^*, \theta)^*$-closed, if the image of each e^*-θ-closed set F of X is (e^*, θ)-closed in Y.

Lemma 7 *A function $f : X \to Y$ is $(e^*, \theta)^*$-closed if and only if for each subset B of Y and each e^*-θ-open set U containing $f^{-1}[B]$, there exists an (e^*, θ)-open set V of Y containing B such that $f^{-1}[V] \subseteq U$.*

Proof It is similar to the proof of Lemma 3. □

Theorem 15 *If $f : X \to Y$ is a $(e^*, \theta)^*$-closed and strongly e^*-irresolute surjection from an $(e^*, \theta)^*$-normal space X onto a space Y, then Y is $(e^*, \theta)^*$-normal.*

Proof Let $K, L \in e^*\theta C(Y)$ and $K \cap L = \emptyset$.

$$\left.\begin{array}{r}(K, L \in e^*\theta C(Y))(K \cap L = \emptyset)\\ f \text{ is strongly } e^*\text{-irresolute}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (f^{-1}[K], f^{-1}[L] \in e^*\theta C(X))(f^{-1}[K] \cap f^{-1}[L] = f^{-1}[K \cap L] = \emptyset) \left.\begin{array}{r}\\ X \text{ is } (e^*, \theta)^*\text{-normal}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U, V \in e^*\theta O(X))(f^{-1}[K] \subseteq U)(f^{-1}[L] \subseteq V)(U \cap V = \emptyset) \left.\begin{array}{r}\\ f \text{ is } (e^*, \theta)^*\text{-closed}\end{array}\right\} \overset{\text{Lemma 7}}{\Rightarrow}$$

$$\Rightarrow (\exists G, H \in (e^*, \theta)O(Y))(K \subseteq G)(L \subseteq H)(f^{-1}[G] \subseteq U)(f^{-1}[H] \subseteq V)(G \cap H = \emptyset).$$

Then, Y is $(e^*, \theta)^*$-normal from Theorem 14 (6). □

Theorem 16 *If $f : X \to Y$ is a pre $e^*\theta$-closed and strongly e^*-irresolute injection and Y is $(e^*, \theta)^*$-normal, then X is $(e^*, \theta)^*$-normal.*

Proof Let $K, L \in e^*\theta C(X)$ and $K \cap L = \emptyset$.

$$\left.\begin{array}{r}(K, L \in e^*\theta C(X))(K \cap L = \emptyset)\\ f \text{ is pre } e^*\theta\text{-closed injection}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (f[K], f[L] \in e^*\theta C(Y))(f[K] \cap f[L] = f[K \cap L] = \emptyset) \left.\begin{array}{r}\\ Y \text{ is } (e^*, \theta)^*\text{-normal}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U, V \in e^*\theta O(Y))(f[K] \subseteq U)(f[L] \subseteq V)(U \cap V = \emptyset) \left.\begin{array}{r}\\ f \text{ is strongly } e^*\text{-irresolute}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (f^{-1}[U], f^{-1}[V] \in e^*\theta O(X))(A \subseteq f^{-1}[U])(B \subseteq f^{-1}[V])(f^{-1}[U] \cap f^{-1}[V] = \emptyset).$$ □

3 Conclusion

Many forms of regular and normal spaces have been studied by many authors in recent years. This paper is concerned with the notion of $e^*\theta$-regular and $e^*\theta$-normal spaces which are defined by utilizing the concept of e^*-θ-open set. We have seen that these concepts are weaker and stronger than many generalized regular and normal space forms in the literature, as will be seen in Remarks 1and 3. We believe that this work will shed light on further studies related to continuity and convergence, etc. known from functional analysis.

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On e^*-Normal Spaces

Serap Erdem, Murad Özkoç, and Takashi Noiri

Abstract The main goal of this chapter is to study the class of e^*-normal spaces (Ekici in Carpathian J Math 24:37–45, 2008 [6]) via the concept of e^*-open set (Ekici in Math Morav 13:29–36, 2009 [7]) defined by Ekici. Also, we investigate the relationships among some other types of normality existing in the literature such as s-normal space (Maheshwari and Prasad in Bull Math Soc Sci Math R.S.Roumanie (N.S.) 22(68):27–29, 1978 [13]), p-normal space (Paul and Bhattacharyya in Soochow J Math 21(3):273–289, 1995 [18]), and β-normal space (Ravi et al. in Appl Math 2(1):27–40, 2015 [19]). Moreover, we introduce and study the class of generalized e^*-closed functions. Finally, we obtain not only characterizations of e^*-normal spaces, but also present some preservation theorems.

1 Introduction

The notion of normality is one of the most important properties in topological spaces. Recently, several forms of notion of normality such as quasi-normal space [5], semi-g-normal space [8] and semi-normal space [16] have been studied by several authors. Also, the notions of s-normal spaces [13], p-normal spaces [18] and β-normal spaces [19] have been introduced and studied by Maheswari and Prasad, Paul and Bhattacharya, Ravi et al. respectively. Moreover, decomposition of the notion of normality [10] have been studied by Kohli and Das.

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The investigations of g-closed sets [12] in topological spaces have been initiated by Levine in 1970. Since then several modifications of g-closed sets have been defined and examined by many authors. In 1990, Arya and Nour introduced the notion of gs-closed sets [2]. In 1996, Maki et al. introduced and studied the notion of gp-closed set [14]. In 2014 and 2016, Ayhan and Özkoç defined and investigated the notions of ge-closed sets [3] and πge-closed sets [4].

On the other hand, one of the most important abstract branches of mathematics that originated from classical analysis is functional analysis. Undoubtedly, the fundamental notion of functional analysis is metric spaces which are similar to that of real line $\mathbb{R}$ in classical analysis. Metric spaces come across as an abstraction of the real line $\mathbb{R}$. The notion of topological space is independent from the notion of distance while the notion of metric space is defined by utilizing the notion of distance. Namely, topological spaces are the abstraction of metric spaces. Moreover, it is well known that not every compact topological space is normal space although all compact metric spaces are normal. Furthermore, a topological vector space is one of the fundamental structures studied in functional analysis. A topological vector space is a vector space that is also a topological space with the property that the vector space operations are also continuous functions. The notion of normal space is highly useful notion in topological vector spaces. Banach spaces, Hilbert spaces, and Sobolev spaces are well-known examples of topological vector spaces.

The aim of this paper is to study the notion of e^*-normal spaces and obtain some of its characterizations. Also, we define the class of generalized e^*-closed functions. Furthermore, we obtain properties of generalized e^*-closed functions and preservation theorems related to e^*-normal spaces.

2 Preliminaries

Throughout this paper, X and Y represent topological spaces. Let X be a topological space and A a subset of X. The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. The family of all closed (resp. open) sets of X is denoted by $C(X)$ (resp. $O(X)$). Recall that a set A is called regular open [20] (resp. regular closed [20]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A subset A of a space X is called e^*-open [7] (resp. β-open [1], preopen [15], semi-open [11]) if $A \subseteq cl(int(\delta\text{-}cl(A)))$ (resp. $A \subseteq cl(int(cl(A)))$, $A \subseteq int(cl(A))$, $A \subseteq cl(int(A))$). The complement of an e^*-open (resp. β-open, preopen, semi-open) set is called e^*-closed [7] (resp. β-closed [1], preclosed [15], semi-closed [11]). The intersection of all e^*-closed (resp. β-closed, regular closed, preclosed, semi-closed) sets of X containing A is called the e^*-closure [7] (resp. β-closure [1], δ-closure [22], preclosure [15], semi-closure [11]) of A and is denoted by $e^*\text{-}cl(A)$ (resp. $\beta\text{-}cl(A)$, $\delta\text{-}cl(A)$, $pcl(A)$, $scl(A)$). The union of all e^*-open (resp. β-open, regular open, preopen, semi-open) sets of X contained in A is called the e^*-interior [7] (resp. β-interior [1], δ-interior [22], preinterior [15], semi-interior [11]) of A and is denoted by $e^*\text{-}int(A)$ (resp. $\beta\text{-}int(A)$, $\delta\text{-}int(A)$, $pint(A)$, $sint(A)$).

The family of all e^*-open (resp. e^*-closed, preopen, semi-open, regular open) subsets of X is denoted by $e^*O(X)$ (resp. $e^*C(X)$, $PO(X)$, $SO(X)$, $RO(X)$). The family of all e^*-open (resp. e^*-closed, preopen, semi-open, regular open) sets of X containing a point x of X is denoted by $e^*O(X, x)$ (resp. $e^*C(X, x)$, $PO(X, x)$, $SO(X, x)$, $RO(X, x)$).

Definition 1 A space X is called:
(1) p-normal [18] if each pair of nonempty disjoint closed sets can be separated by disjoint preopen sets.
(2) s-normal [13] if each pair of nonempty disjoint closed sets can be separated by disjoint semi-open sets.
(3) β-normal [19] if each pair of nonempty disjoint closed sets can be separated by disjoint β-open sets.

Definition 2 A subset A of a space X is called:
(1) generalized closed [12] (briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open. The complement of a generalized closed set is called generalized open (briefly g-open). The family of all g-closed (resp. g-open) subsets of X will be denoted by $gC(X)$ (resp. $gO(X)$).
(2) generalized β-closed [21] (briefly $g\beta$-closed) if β-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open. The complement of a generalized β-closed set is called generalized β-open (briefly $g\beta$-open).
(3) β-generalized closed [19] (briefly βg-closed) if β-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is β-open. The complement of a β-generalized closed set is called β-generalized open (briefly βg-open).

Definition 3 ([7]) A subset A of a space X is called e^*-regular if it is e^*-open and e^*-closed. The family of all e^*-regular subsets of X will be denoted by $e^*R(X)$.

Lemma 1 ([9]) *Let A be a subset of a space X. Then, $A \in e^*O(X)$ if and only if e^*-$cl(A) \in e^*R(X)$.*

Lemma 2 ([7]) *For any subset A of a space X, the followings hold:*
(1) e^*-$int(A) \in e^*O(X)$,
(2) e^*-$cl(A) \in e^*C(X)$,
(3) e^*-$cl(\backslash A) = \backslash e^*$-$int(A)$,
(4) e^*-$cl(e^*$-$cl(A)) = e^*$-$cl(A)$.

3 e^*-Normal Spaces

Definition 4 A topological space X is said to be e^*-normal [6] if each pair of nonempty disjoint closed sets can be separated by disjoint e^*-open sets.

Remark 1 From Definitions 1 and 4, we have the following diagram. The converses of these implications are not true in general as shown by the following example.

$$\begin{array}{ccccc} p\text{-normal} & \rightarrow & \beta\text{-normal} & \rightarrow & e^*\text{-normal} \\ \uparrow & & \uparrow & & \\ \text{normal} & \rightarrow & s\text{-normal} & & \end{array}$$

Example 1 Let $X := \{a, b, c, d, e\}$ and $\tau := \{\emptyset, X, \{b\}, \{a, b\}, \{a, b, c, e\}, \{b, c, d, e\}, \{b, c, e\}\}$. Then, the space X is e^*-normal but not β-normal.

Theorem 1 *Let (X, τ) be a topological space. Then, the following statements are equivalent:*
(1) X is e^*-normal;
(2) For every pair of open sets U and V whose union is X, there exist e^*-closed sets A and B such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$;
(3) For every closed set F and every open set K containing F, there exists an e^*-open set U such that $F \subseteq U \subseteq e^*\text{-}cl(U) \subseteq K$.

Proof (1) $\Rightarrow$ (2) : Let $U, V \in O(X)$ and $U \cup V = X$.

$$\left.\begin{array}{r}(U, V \in O(X))(U \cup V = X) \Rightarrow (\backslash U, \backslash V \in C(X))((\backslash U) \cap (\backslash V) = \emptyset) \\ X \text{ is } e^*\text{-normal}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists M, N \in e^*O(X))(\backslash U \subseteq M)(\backslash V \subseteq N)(M \cap N = \emptyset) \\ (A := \backslash M)(B := \backslash N)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (A, B \in e^*C(X))(A \subseteq U)(B \subseteq V)(A \cup B = X).$$

(2) $\Rightarrow$ (3) : Let $F \in C(X)$, $K \in O(X)$ and $F \subseteq K$.

$$\left.\begin{array}{r}(F \in C(X))(K \in O(X))(F \subseteq K) \Rightarrow (\backslash F, K \in O(X))((\backslash F) \cup K = X) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists A, B \in e^*C(X))(A \subseteq \backslash F)(B \subseteq K)(A \cup B = X)$$

$$\left.\begin{array}{r}\Rightarrow (\backslash A, \backslash B \in e^*O(X))(F \subseteq \backslash A)(\backslash K \subseteq \backslash B)((\backslash A) \cap (\backslash B) = \emptyset) \\ (U := \backslash A)(V := \backslash B)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (U, V \in e^*O(X))(F \subseteq U)(\backslash K \subseteq V)(U \cap V = \emptyset)$$

$$\Rightarrow (U \in e^*O(X))(F \subseteq U \subseteq e^*\text{-}cl(U) \subseteq K).$$

(3) $\Rightarrow$ (1) : Let $A, B \in C(X)$ and $A \cap B = \emptyset$.

$$\left.\begin{array}{r}(A, B \in C(X))(A \cap B = \emptyset) \\ K := \backslash B\end{array}\right\} \Rightarrow \left.\begin{array}{r}(A \in C(X))(K \in O(X))(A \subseteq K) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in e^*O(X))(A \subseteq U \subseteq e^*\text{-}cl(U) \subseteq K)$$

$$\left.\begin{array}{r}\Rightarrow (\exists U \in e^*O(X))(B = \backslash K \subseteq \backslash e^*\text{-}cl(U)) \\ V := \backslash e^*\text{-}cl(U)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (U, V \in e^*O(X))(A \subseteq U)(B \subseteq V)(U \cap V = \emptyset). \qquad \square$$

4 The Related Functions with e^*-Normal Spaces

Definition 5 A function $f : (X, \tau) \to (Y, \sigma)$ is called:
(1) pre e^*-open if $f[U]$ is e^*-open in Y for each e^*-open set U of X,
(2) pre e^*-closed if $f[U]$ is e^*-closed in Y for each e^*-closed set U of X,
(3) almost e^*-irresolute if for each $x \in X$ and each e^*-open set V containing $f(x)$ such that $e^*\text{-}cl(f^{-1}[V]) \in e^*O(X, x)$,
(4) e^*-closed [17] if $f[A]$ is e^*-closed in Y for each closed set A of X.

Theorem 2 *A function $f : (X, \tau) \to (Y, \sigma)$ is pre e^*-closed if and only if for each subset A in Y and for each e^*-open set U in X containing $f^{-1}[A]$, there exists an e^*-open set V of Y containing A such that $f^{-1}[V] \subseteq U$.*

Proof *Necessity.* Let $A \subseteq Y$, $U \in e^*O(X)$ and $f^{-1}[A] \subseteq U$.

$$\left.\begin{array}{r} U \in e^*O(X) \Rightarrow \backslash U \in e^*C(X) \\ f \text{ is pre } e^*\text{-closed} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f[\backslash U] \in e^*C(Y) \\ V := \backslash f[\backslash U] \end{array}\right\} \Rightarrow$$
$$\Rightarrow (V \in e^*O(Y))(A \subseteq V)(f^{-1}[V] \subseteq U).$$

Sufficiency. Let $F \in e^*C(X)$.
$$F \in e^*C(X) \Rightarrow (\backslash F \in e^*O(X))(f^{-1}[\backslash f[F]] = \backslash f^{-1}[f[F]] \subseteq \backslash F) \left.\begin{array}{r} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists V \in e^*O(Y))(\backslash f[F] \subseteq V)(f^{-1}[V] \subseteq \backslash F)$$
$$\Rightarrow (\backslash V \in e^*C(Y))(\backslash V \subseteq f[F])(F \subseteq \backslash f^{-1}[V] = f^{-1}[\backslash V])$$
$$\Rightarrow (\backslash V \in e^*C(Y))(\backslash V \subseteq f[F] \subseteq f[f^{-1}[\backslash V]] \subseteq \backslash V)$$
$$\Rightarrow (\backslash V \in e^*C(Y))(\backslash V = f[F])$$
$$\Rightarrow f[F] \in e^*C(Y). \qquad \square$$

Lemma 3 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, the following statements are equivalent:*
(1) f is almost e^*-irresolute;
(2) $f^{-1}[V] \subseteq e^*\text{-}int(e^*\text{-}cl(f^{-1}[V]))$ for each e^*-open subset V of Y.

Proof (1) $\Rightarrow$ (2) : Let $V \in e^*O(Y)$ and $x \in f^{-1}[V]$.
$$(V \in e^*O(Y))(x \in f^{-1}[V]) \Rightarrow V \in e^*O(Y, f(x)) \left.\begin{array}{r} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow e^*\text{-}cl(f^{-1}[V]) \in e^*O(X, x)$$
$$\Rightarrow x \in e^*\text{-}int(e^*\text{-}cl(f^{-1}[V])).$$

(2) $\Rightarrow$ (1) : Let $x \in X$ and $V \in e^*O(Y, f(x))$.
$$V \in e^*O(Y, f(x)) \left.\begin{array}{r} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f^{-1}[V] \subseteq e^*\text{-}int(e^*\text{-}cl(f^{-1}[V])) \subseteq e^*\text{-}cl(f^{-1}[V])$$
$$\Rightarrow e^*\text{-}cl(f^{-1}[V]) \subseteq e^*\text{-}cl(e^*\text{-}int(e^*\text{-}cl(f^{-1}[V]))) \subseteq e^*\text{-}cl(e^*\text{-}cl(f^{-1}[V])) = e^*\text{-}cl(f^{-1}[V])$$
$$\Rightarrow e^*\text{-}cl(f^{-1}[V]) = e^*\text{-}cl(e^*\text{-}int(e^*\text{-}cl(f^{-1}[V]))) \left.\begin{array}{r} \\ \text{Lemma 1} \end{array}\right\} \Rightarrow e^*\text{-}cl(f^{-1}[V]) \in e^*O(X, x). \qquad \square$$

Theorem 3 *A bijective function $f : (X, \tau) \to (Y, \sigma)$ is almost e^*-irresolute if and only if $f[e^*\text{-}cl(U)] \subseteq e^*\text{-}cl(f[U])$ for each e^*-open subset U of X.*

Proof *Necessity.* Let $U \in e^*O(X)$ and $y \notin e^*\text{-}cl(f[U])$.
$y \notin e^*\text{-}cl(f[U]) \Rightarrow (\exists V \in e^*O(Y, y))(V \cap f[U] = \emptyset)$
$\Rightarrow (\exists V \in e^*O(Y, y))(f^{-1}[V] \cap U \subseteq f^{-1}[V] \cap f^{-1}[f[U]] = f^{-1}[V \cap f[U]] = \emptyset)$
$\Rightarrow (\exists V \in e^*O(Y, y))(f^{-1}[V] \cap U = \emptyset)$
$\Rightarrow (\exists V \in e^*O(Y, y))(e^*\text{-}int(e^*\text{-}cl(f^{-1}[V])) \cap e^*\text{-}cl(U) = \emptyset)$ } Lemma 3 $\Rightarrow$
$\Rightarrow (\exists V \in e^*O(Y, y))(f^{-1}[V] \cap e^*\text{-}cl(U) = \emptyset)$
$\Rightarrow (\exists V \in e^*O(Y, y))(f[f^{-1}[V] \cap e^*\text{-}cl(U)] = \emptyset)$ } f is bijective $\Rightarrow$
$\Rightarrow (\exists V \in e^*O(Y, y))(V \cap f[e^*\text{-}cl(U)] = \emptyset)$
$\Rightarrow y \notin f[e^*\text{-}cl(U)]$.

Sufficiency. Let $V \in e^*O(Y)$.

$V \in e^*O(Y)$, $U := \backslash e^*\text{-}cl(f^{-1}[V])$ } $\Rightarrow$ $U \in e^*O(X)$ Hypothesis } $\Rightarrow f[e^*\text{-}cl(U)] \subseteq e^*\text{-}cl(f[U])$
$\Rightarrow e^*\text{-}cl(U) \subseteq f^{-1}[f[e^*\text{-}cl(U)]] \subseteq f^{-1}[e^*\text{-}cl(f[U])]$
$\Rightarrow \backslash e^*\text{-}int(e^*\text{-}cl(f^{-1}[V])) = e^*\text{-}cl(U) \subseteq f^{-1}[e^*\text{-}cl(f[U])] \subseteq \backslash f^{-1}[V]$
$\Rightarrow f^{-1}[V] \subseteq e^*\text{-}int(e^*\text{-}cl(f^{-1}[V]))$
Then, f is almost e^*-irresolute from Lemma 3. □

Theorem 4 *If $f : (X, \tau) \to (Y, \sigma)$ is a pre e^*-closed continuous function from an e^*-normal space X onto a space Y, then Y is an e^*-normal space.*

Proof Let $M_1, M_2 \in C(Y)$ and $M_1 \cap M_2 = \emptyset$.
$(M_1, M_2 \in C(Y))(M_1 \cap M_2 = \emptyset)$ } f is continuous $\Rightarrow$
$\Rightarrow (f^{-1}[M_1], f^{-1}[M_2] \in C(X))(f^{-1}[M_1] \cap f^{-1}[M_2] = f^{-1}[M_1 \cap M_2] = \emptyset)$ } X is e^*-normal $\Rightarrow$
$\Rightarrow (\exists U, V \in e^*O(X))(f^{-1}[M_1] \subseteq U)(f^{-1}[M_2] \subseteq V)(U \cap V = \emptyset)$ } f is pre e^*-closed $\overset{\text{Theorem 2}}{\Rightarrow}$
$\Rightarrow (\exists A, B \in e^*O(Y))(M_1 \subseteq A)(M_2 \subseteq B)(f^{-1}[A] \subseteq U)(f^{-1}[B] \subseteq V)(U \cap V = \emptyset)$
$\Rightarrow (\exists A, B \in e^*O(Y))(M_1 \subseteq A)(M_2 \subseteq B)(A \cap B = \emptyset)$. □

Theorem 5 *If $f : (X, \tau) \to (Y, \sigma)$ is an e^*-closed continuous surjection and X is normal, then Y is e^*-normal.*

Proof Let $A, B \in C(Y)$ and $A \cap B = \emptyset$.
$(A, B \in C(Y))(A \cap B = \emptyset)$ } f is continuous $\Rightarrow$
$\Rightarrow (f^{-1}[A], f^{-1}[B] \in C(X))(f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B] = \emptyset)$ } X is normal $\Rightarrow$
$\Rightarrow (\exists U, V \in O(X))(f^{-1}[A] \subseteq U)(f^{-1}[B] \subseteq V)(U \cap V = \emptyset)$
$(M_1 := \backslash f[\backslash U])(M_2 := \backslash f[\backslash V])$
f is an e^*-closed surjection } $\Rightarrow$
$\Rightarrow (M_1, M_2 \in e^*O(Y))(A \subseteq M_1)(B \subseteq M_2)(M_1 \cap M_2 = \emptyset)$. □

5 Generalized e^*-Closed Functions

Definition 6 Let (X, τ) be a topological space. A subset A of X is called generalized e^*-closed (briefly ge^*-closed) if $e^*\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open. The complement of a generalized e^*-closed set is called generalized e^*-open (briefly ge^*-open). The family of all ge^*-closed (resp. ge^*-open) subsets of X will be denoted by $ge^*C(X)$ (resp. $ge^*O(X)$).

Definition 7 Let (X, τ) be a topological space. A subset A of X is called e^*-generalized closed (briefly e^*g-closed) if $e^*\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is e^*-open. The complement of an e^*-generalized closed set is called e^*-generalized open (briefly e^*g-open). The family of all e^*g-closed (resp. e^*g-open) subsets of X will be denoted by $e^*gC(X)$ (resp. $e^*gO(X)$).

Remark 2 From Definitions 6 and 7, we have the following diagram.

$$\begin{array}{ccccc}
\text{closed} & \rightarrow & \beta\text{-closed} & \rightarrow & e^*\text{-closed} \\
 & & \downarrow & & \downarrow \\
\downarrow & & \beta g\text{-closed} & & e^*g\text{-closed} \\
 & & \downarrow & & \downarrow \\
g\text{-closed} & \rightarrow & g\beta\text{-closed} & \rightarrow & ge^*\text{-closed}
\end{array}$$

However, none of these implications is reversible as shown by the following examples.

Example 2 Let $X := \{a, b, c, d, e\}$ and $\tau := \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then:
(1) the set $\{a, c\}$ is ge^*-closed but it is not $g\beta$-closed.
(2) the set $\{a, c, d, e\}$ is ge^*-closed but it is not e^*g-closed.

Definition 8 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called:
(1) e^*g-closed if $f[A]$ is e^*g-closed in Y for each closed set A of X.
(2) ge^*-closed if $f[A]$ is ge^*-closed in Y for each closed set A of X.

Definition 9 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called:
(1) quasi e^*-closed if $f[A]$ is closed in Y for each e^*-closed set A of X.
(2) e^*-e^*g-closed if $f[A]$ is e^*g-closed in Y for each e^*-closed set A of X.
(3) e^*-ge^*-closed if $f[A]$ is ge^*-closed in Y for each e^*-closed set A of X.
(4) almost ge^*-closed if $f[A]$ is ge^*-closed in Y for each regular closed set A of X.

Remark 3 The following diagram holds for a function $f : X \rightarrow Y$.

$$\begin{array}{ccccc}
e^*\text{-}ge^*\text{-closed} & \rightarrow & ge^*\text{-closed} & \rightarrow & \text{almost } ge^*\text{-closed} \\
\uparrow & & \uparrow & & \uparrow \\
e^*\text{-}e^*g\text{-closed} & \rightarrow & e^*g\text{-closed} & & \text{almost } g\beta\text{-closed} \\
\downarrow & & \uparrow & & \\
e^*\text{-closed} & \leftarrow & \text{pre } e^*\text{-closed} & & \\
\uparrow & & \uparrow & & \\
\text{closed} & \leftarrow & \text{quasi } e^*\text{-closed} & &
\end{array}$$

Definition 10 A function $f : (X, \tau) \to (Y, \sigma)$ is called e^*-ge^*-continuous if $f^{-1}[F]$ is ge^*-closed in X for every e^*-closed set F of Y.

Definition 11 A function $f : (X, \tau) \to (Y, \sigma)$ is called e^*-irresolute [6] if $f^{-1}[V]$ is e^*-open in X for every e^*-open set V of Y.

Remark 4 Every e^*-irresolute function is e^*-ge^*-continuous but not conversely.

Theorem 6 *If $f : (X, \tau) \to (Y, \sigma)$ is continuous e^*-ge^*-closed, then $f[A]$ is ge^*-closed in Y for every ge^*-closed set A of X.*

Proof Let $A \in ge^*C(X)$, $U \in O(Y)$ and $f[A] \subseteq U$.

$$\left.\begin{array}{r}(U \in O(Y))(f[A] \subseteq U)\\ f \text{ is continuous}\end{array}\right\} \Rightarrow (f^{-1}[U] \in O(X))(A \subseteq f^{-1}[f[A]] \subseteq f^{-1}[U]) \left.\begin{array}{r}\\ A \in ge^*C(X)\end{array}\right\} \Rightarrow$$

$$\Rightarrow e^*\text{-}cl(A) \subseteq f^{-1}[U] \Rightarrow f[e^*\text{-}cl(A)] \subseteq f[f^{-1}[U]] \subseteq U \left.\begin{array}{r}\\ f \text{ is } e^*\text{-}ge^*\text{-closed}\end{array}\right\} \Rightarrow$$

$$\Rightarrow e^*\text{-}cl(f[A]) \subseteq e^*\text{-}cl(f[e^*\text{-}cl(A)]) \subseteq U. \qquad \square$$

Theorem 7 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be any two functions. Then:*
(1) if f is e^*-ge^*-closed and g is continuous e^*-ge^*-closed, then $g \circ f$ is e^*-ge^*-closed.
(2) if f is pre e^*-closed and g is e^*-ge^*-closed, then $g \circ f$ is e^*-ge^*-closed.

Proof (1) Let $A \in e^*C(X)$, $U \in O(Z)$ and $(g \circ f)[A] \subseteq U$.

$$\left.\begin{array}{r}(U \in O(Z))((g \circ f)[A] \subseteq U)\\ g \text{ is continuous}\end{array}\right\} \Rightarrow (g^{-1}[U] \in O(Y))(g^{-1}[(g \circ f)[A]] \subseteq g^{-1}[U])$$

$$\left.\begin{array}{r}\Rightarrow (g^{-1}[U] \in O(Y))(f[A] \subseteq g^{-1}[(g \circ f)[A]] \subseteq g^{-1}[U])\\ (A \in e^*C(X))(f \text{ is } e^*\text{-}ge^*\text{-closed}) \Rightarrow f[A] \in ge^*C(Y)\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow e^*\text{-}cl(f[A]) \subseteq g^{-1}[U] \Rightarrow g[e^*\text{-}cl(f[A])] \subseteq g[g^{-1}[U]] \subseteq U\\ g \text{ is } e^*\text{-}ge^*\text{-closed}\end{array}\right\} \Rightarrow$$

$$\Rightarrow e^*\text{-}cl((g \circ f)[A]) \subseteq e^*\text{-}cl(g[e^*\text{-}cl(f[A])]) \subseteq U.$$

(2) Let $A \in e^*C(X)$.

$$\left.\begin{array}{r}A \in e^*C(X)\\ f \text{ is pre } e^*\text{-closed}\end{array}\right\} \Rightarrow \left.\begin{array}{r}f[A] \in e^*C(Y)\\ g \text{ is } e^*\text{-}ge^*\text{-closed}\end{array}\right\} \Rightarrow g[f[A]] = (g \circ f)[A] \in ge^*C(Z).$$

$\square$

Theorem 8 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be any two functions. If $g \circ f$ is e^*-ge^*-closed and f is an e^*-irresolute surjection, then g is e^*-ge^*-closed.*

Proof Let $A \in e^*C(Y)$.

$$\left.\begin{array}{r}A \in e^*C(Y) \Rightarrow \backslash A \in e^*O(Y)\\ f \text{ is } e^*\text{-irresolute}\end{array}\right\} \Rightarrow f^{-1}[\backslash A] = \backslash f^{-1}[A] \in e^*O(X)$$

$$\left.\begin{array}{r}\Rightarrow f^{-1}[A] \in e^*C(X)\\ f \text{ is surjective}\\ g \circ f \text{ is } e^*\text{-}ge^*\text{-closed}\end{array}\right\} \Rightarrow (g \circ f)[f^{-1}[A]] = g[A] \in ge^*C(Z). \qquad \square$$

Lemma 4 *Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective function. Then:*
(1) f is ge^*-closed if and only if for each subset B of Y and each open subset U of X containing $f^{-1}[B]$, there exists a ge^*-open set V of Y such that $B \subseteq V$ and $f^{-1}[V] \subseteq U$.
(2) f is e^*-ge^*-closed if and only if for each subset B of Y and each e^*-open subset U of X containing $f^{-1}[B]$, there exists a ge^*-open set V of Y such that $B \subseteq V$ and $f^{-1}[V] \subseteq U$.

Proof (1) *Necessity.* Let $B \subseteq Y,\ U \in O(X)$ and $f^{-1}[B] \subseteq U$.

$$\left.\begin{array}{r}(U \in O(X))(f^{-1}[B] \subseteq U) \Rightarrow (\backslash U \in C(X))(\backslash U \subseteq \backslash f^{-1}[B] = f^{-1}[\backslash B]) \\ f \text{ is a } ge^*\text{-closed surjection}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (f[\backslash U] \in ge^*C(Y))(f[\backslash U] \subseteq f[f^{-1}[\backslash B]] = \backslash B) \\ V := \backslash f[\backslash U]\end{array}\right\} \Rightarrow$$

$$\Rightarrow (V \in ge^*O(Y))(B \subseteq V)(f^{-1}[V] \subseteq U).$$

Sufficiency. Let $A \in C(X),\ K \in O(Y)$ and $f[A] \subseteq K$.

$$\left.\begin{array}{r}(K \in O(Y))(A \in C(X))(f[A] \subseteq K) \Rightarrow (\backslash A \in O(X))(f^{-1}[\backslash K] \subseteq \backslash A) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists V \in ge^*O(Y))(\backslash K \subseteq V)(f^{-1}[V] \subseteq \backslash A)$$

$$\left.\begin{array}{r}\Rightarrow (\backslash V \in ge^*C(Y))(\backslash V \subseteq K)(A \subseteq \backslash f^{-1}[V] = f^{-1}[\backslash V]) \\ f \text{ is surjective}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (e^*\text{-}cl(\backslash V) \subseteq K)(f[A] \subseteq \backslash V)$$
$$\Rightarrow e^*\text{-}cl(f[A]) \subseteq e^*\text{-}cl(\backslash V) \subseteq K.$$

(2) It is done similarly to the proof of (1). □

Theorem 9 *A function $f : (X, \tau) \to (Y, \sigma)$ is e^*-ge^*-continuous if and only if $f^{-1}[V]$ is ge^*-open in X for every V is e^*-open in Y.*

Proof It is straightforward. □

Theorem 10 *If $f : (X, \tau) \to (Y, \sigma)$ is closed e^*-ge^*-continuous, then $f^{-1}[A]$ is ge^*-closed in X for every ge^*-closed set A of Y.*

Proof Let $A \in ge^*C(Y),\ U \in O(X)$ and $f^{-1}[A] \subseteq U$.

$$\left.\begin{array}{r}(U \in O(X))(f^{-1}[A] \subseteq U) \\ V := \backslash f[\backslash U] \\ f \text{ is closed}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(V \in O(Y))(A \subseteq V)(f^{-1}[V] \subseteq U) \\ A \in ge^*C(Y)\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (e^*\text{-}cl(A) \subseteq V)(f^{-1}[V] \subseteq U) \Rightarrow f^{-1}[A] \subseteq f^{-1}[e^*\text{-}cl(A)] \subseteq f^{-1}[V] \subseteq U \\ f \text{ is } e^*\text{-}ge^*\text{-continuous}\end{array}\right\} \Rightarrow$$

$$\Rightarrow e^*\text{-}cl(f^{-1}[A]) \subseteq e^*\text{-}cl(f^{-1}[e^*\text{-}cl(A)]) \subseteq U.$$ □

Corollary 1 *If $f : (X, \tau) \to (Y, \sigma)$ is closed e^*-irresolute, then $f^{-1}[A]$ is ge^*-closed in X for every ge^*-closed set A of Y.*

Proof From Theorem 10, it is obvious since every e^*-irresolute implies e^*-ge^*-continuous. □

Theorem 11 *If $f : (X, \tau) \to (Y, \sigma)$ is an open e^*-ge^*-continuous bijection, then $f^{-1}[A]$ is ge^*-closed in X for every ge^*-closed set A of Y.*

Proof Let $A \in ge^*C(Y)$, $U \in O(X)$ and $f^{-1}[A] \subseteq U$.

$$\left.\begin{array}{r}(U \in O(X))(f^{-1}[A] \subseteq U)\\ f \text{ is an open surjection}\end{array}\right\} \Rightarrow (f[U] \in O(Y))(A = f[f^{-1}[A]] \subseteq f[U]) \left.\begin{array}{r}\\ A \in ge^*C(Y)\end{array}\right\} \Rightarrow$$

$$\Rightarrow e^*\text{-}cl(A) \subseteq f[U] \left.\begin{array}{r}\\ f \text{ is injective}\end{array}\right\} \Rightarrow f^{-1}[A] \subseteq f^{-1}[e^*\text{-}cl(A)] \subseteq f^{-1}[f[U]] = U \left.\begin{array}{r}\\ f \text{ is } e^*\text{-}ge^*\text{-continuous}\end{array}\right\} \Rightarrow$$

$$\Rightarrow e^*\text{-}cl(f^{-1}[A]) \subseteq e^*\text{-}cl(f^{-1}[e^*\text{-}cl(A)]) \subseteq U. \qquad \square$$

Theorem 12 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be any two functions. If $g \circ f$ is e^*-ge^*-closed and g is an open e^*-ge^*-continuous bijection, then f is e^*-ge^*-closed.*

Proof Let $A \in e^*C(X)$.

$$\left.\begin{array}{r}A \in e^*C(X)\\ g \circ f \text{ is } e^*\text{-}ge^*\text{-closed}\end{array}\right\} \Rightarrow \left.\begin{array}{c}(g \circ f)[A] \in ge^*C(Z)\\ g \text{ is an open } e^* - ge^* - \text{continuous bijection}\end{array}\right\} \overset{\text{Theorem 11}}{\Rightarrow}$$

$$\Rightarrow g^{-1}[(g \circ f)[A]] = f[A] \in ge^*C(Y). \qquad \square$$

Theorem 13 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be any two functions. If $g \circ f$ is e^*-ge^*-closed and g is a closed e^*-ge^*-continuous injection, then f is e^*-ge^*-closed.*

Proof Let $A \in e^*C(X)$.

$$\left.\begin{array}{r}A \in e^*C(X)\\ g \circ f \text{ is } e^*\text{-}ge^*\text{-closed}\end{array}\right\} \Rightarrow \left.\begin{array}{c}(g \circ f)[A] \in ge^*C(Z)\\ g \text{ is a closed } e^*\text{-}ge^*\text{-continuous injection}\end{array}\right\} \overset{\text{Theorem 10}}{\Rightarrow}$$

$$\Rightarrow g^{-1}[(g \circ f)[A]] = f[A] \in ge^*C(Y). \qquad \square$$

6 Preservation Theorems and Other Characterizations of *e**-Normal Spaces

In this section, we obtain the characterizations of e^*-normal spaces and some preservation theorems.

Theorem 14 *For a topological space X, the following statements are equivalent:*
(1) X is e^*-normal;
(2) For any pair of disjoint closed sets A and B of X, there exist disjoint ge^*-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$;
(3) For each closed set A and any open set B containing A, there exists a ge^*-open set U such that $A \subseteq U \subseteq e^*\text{-}cl(U) \subseteq B$;
(4) For each closed set A and any g-open set B containing A, there exists an e^*-open

set U such that $A \subseteq U \subseteq e^*\text{-}cl(U) \subseteq int(B)$;
(5) For each closed set A and any g-open set B containing A, there exists a ge^*-open set U such that $A \subseteq U \subseteq e^*\text{-}cl(U) \subseteq int(B)$;
(6) For each g-closed set A and any open set B containing A, there exists an e^*-open set U such that $cl(A) \subseteq U \subseteq e^*\text{-}cl(U) \subseteq B$;
(7) For each g-closed set A and any open set B containing A, there exists a ge^*-open set U such that $cl(A) \subseteq U \subseteq e^*\text{-}cl(U) \subseteq B$.

Proof (1) $\Rightarrow$ (2) : Let $A, B \in C(X)$ and $A \cap B = \emptyset$.

$$\left.\begin{array}{r}(A, B \in C(X))(A \cap B = \emptyset)\\ X \text{ is } e^*\text{-normal}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U, V \in e^*O(X))(A \subseteq U)(B \subseteq V)(U \cap V = \emptyset) \left.\begin{array}{r} \\ e^*O(X) \subseteq ge^*O(X)\end{array}\right\} \Rightarrow$$
$$\Rightarrow (U, V \in ge^*O(X))(A \subseteq U)(B \subseteq V)(U \cap V = \emptyset).$$

(2) $\Rightarrow$ (3) : Let $A \in C(X),\ B \in O(X)$ and $A \subseteq B$.

$$(A \in C(X))(B \in O(X))(A \subseteq B) \Rightarrow (A, \backslash B \in C(X))(A \cap (\backslash B) = \emptyset) \left.\begin{array}{r} \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U, V \in ge^*O(X))(A \subseteq U)(\backslash B \subseteq V)(U \cap V = \emptyset)$$
$$\Rightarrow (U \in ge^*O(X))(\backslash V \in ge^*C(X))(A \subseteq U)(\backslash V \subseteq B)(U \cap V = \emptyset)$$
$$\Rightarrow (U \in ge^*O(X))(A \subseteq U)(e^*\text{-}cl(\backslash V) \subseteq B)(U \subseteq \backslash V)$$
$$\Rightarrow (U \in ge^*O(X))(A \subseteq U \subseteq e^*\text{-}cl(U) \subseteq e^*\text{-}cl(\backslash V) \subseteq B).$$

(3) $\Rightarrow$ (4) : Let $A \in C(X),\ B \in gO(X)$ and $A \subseteq B$.

$$(A \in C(X))(B \in gO(X))(A \subseteq B) \Rightarrow (\backslash A \in O(X))(\backslash B \in gC(X))(\backslash B \subseteq \backslash A)$$
$$\Rightarrow (A \in C(X))(cl(\backslash B) \subseteq \backslash A) \Rightarrow (A \in C(X))(int(B) \in O(X))(A \subseteq int(B)) \left.\begin{array}{r} \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in ge^*O(X))(A \subseteq U \subseteq e^*\text{-}cl(U) \subseteq int(B))$$
$$\Rightarrow (\backslash U \in ge^*C(X))(\backslash int(B) \subseteq \backslash e^*\text{-}cl(U) \subseteq \backslash U \subseteq \backslash A)$$
$$\Rightarrow \backslash int(B) \subseteq \backslash e^*\text{-}cl(U) \subseteq \backslash U \subseteq e^*\text{-}cl(\backslash U) \subseteq \backslash A$$
$$\Rightarrow A \subseteq e^*\text{-}int(U) \subseteq U \subseteq e^*\text{-}cl(U) \subseteq int(B) \left.\begin{array}{r} \\ V := e^*\text{-}int(U)\end{array}\right\} \Rightarrow$$
$$\Rightarrow (V \in e^*O(X))(A \subseteq V \subseteq e^*\text{-}cl(V) \subseteq int(B)).$$

(4) $\Rightarrow$ (5) : Let $A \in C(X),\ B \in gO(X)$ and $A \subseteq B$.

$$\left.\begin{array}{r}(A \in C(X))(B \in gO(X))(A \subseteq B)\\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in e^*O(X))(A \subseteq U \subseteq e^*\text{-}cl(U) \subseteq int(B)) \left.\begin{array}{r} \\ e^*O(X) \subseteq ge^*O(X)\end{array}\right\} \Rightarrow$$
$$\Rightarrow (U \in ge^*O(X))(A \subseteq U \subseteq e^*\text{-}cl(U) \subseteq int(B)).$$

(5) $\Rightarrow$ (6) : Let $A \in gC(X),\ B \in O(X)$ and $A \subseteq B$.

$$(A \in gC(X))(B \in O(X))(A \subseteq B) \Rightarrow (\backslash A \in gO(X))(\backslash B \in C(X))(\backslash B \subseteq \backslash A) \left.\begin{array}{r} \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in ge^*O(X))(\backslash B \subseteq U \subseteq e^*\text{-}cl(U) \subseteq int(\backslash A))$$

$$\Rightarrow (\backslash U \in ge^*C(X))(cl(A) \subseteq \backslash e^*\text{-}cl(U) \subseteq \backslash U \subseteq B)$$
$$\left.\begin{array}{r} \Rightarrow cl(A) \subseteq \backslash e^*\text{-}cl(U) \subseteq \backslash U \subseteq e^*\text{-}cl(\backslash U) \subseteq B \\ V := \backslash e^*\text{-}cl(U) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (V \in e^*O(X))(cl(A) \subseteq V \subseteq e^*\text{-}cl(V) \subseteq B).$$

(6) $\Rightarrow$ (7) : Let $A \in gC(X),\ B \in O(X)$ and $A \subseteq B$.

$$\left.\begin{array}{r} (A \in gC(X))(B \in O(X))(A \subseteq B) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (\exists U \in e^*O(X))(cl(A) \subseteq U \subseteq e^*\text{-}cl(U) \subseteq B) \\ e^*O(X) \subseteq ge^*O(X) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (U \in ge^*O(X))(cl(A) \subseteq U \subseteq e^*\text{-}cl(U) \subseteq B).$$

(7) $\Rightarrow$ (1) : Let $A \in C(X),\ B \in O(X)$ and $A \subseteq B$.

$$\left.\begin{array}{r} (A \in C(X))(B \in O(X))(A \subseteq B) \\ C(X) \subseteq gC(X) \end{array}\right\} \Rightarrow \left.\begin{array}{r} (A \in gC(X))(B \in O(X))(A \subseteq B) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in ge^*O(X))(cl(A) \subseteq U \subseteq e^*\text{-}cl(U) \subseteq B)$$
$$\Rightarrow (\backslash U \in ge^*C(X))(\backslash B \subseteq \backslash e^*\text{-}cl(U) \subseteq \backslash U \subseteq \backslash cl(A))$$
$$\Rightarrow \backslash B \subseteq \backslash e^*\text{-}cl(U) \subseteq \backslash U \subseteq e^*\text{-}cl(\backslash U) \subseteq \backslash cl(A)$$
$$\Rightarrow A \subseteq cl(A) \subseteq e^*\text{-}int(U) \subseteq U \subseteq e^*\text{-}cl(U) \subseteq B$$
$$\left.\begin{array}{r} \Rightarrow A \subseteq e^*\text{-}int(U) \subseteq e^*\text{-}cl(e^*\text{-}int(U)) \subseteq e^*\text{-}cl(U) \subseteq B \\ V := e^*\text{-}int(U) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (V \in e^*O(X))(A \subseteq V \subseteq e^*\text{-}cl(V) \subseteq B). \qquad \square$$

Theorem 15 *If $f : (X, \tau) \to (Y, \sigma)$ is a continuous quasi e^*-closed surjection and X is e^*-normal, then Y is normal.*

Proof Let $A, B \in C(Y)$ and $A \cap B = \emptyset$.

$$\left.\begin{array}{r} (A, B \in C(Y))(A \cap B = \emptyset) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (f^{-1}[A], f^{-1}[B] \in C(X))(f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B] = \emptyset) \\ X \text{ is } e^*\text{-normal} \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (\exists U_1, U_2 \in e^*O(X))(f^{-1}[A] \subseteq U_1)(f^{-1}[B] \subseteq U_2)(U_1 \cap U_2 = \emptyset) \\ (V_1 := \backslash f[\backslash U_1])(V_2 := \backslash f[\backslash U_2]) \\ f \text{ is a quasi } e^*\text{-closed surjection} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (V_1, V_2 \in O(Y))(A \subseteq V_1)(B \subseteq V_2)(V_1 \cap V_2 = \emptyset). \qquad \square$$

Lemma 5 *A subset A of a space X is ge^*-open if and only if $F \subseteq e^*\text{-}int(A)$ whenever F is closed in X and $F \subseteq A$.*

Proof It is straightforward. $\square$

Theorem 16 *If $f : (X, \tau) \to (Y, \sigma)$ is a closed e^*-ge^*-continuous injection and Y is e^*-normal, then X is e^*-normal.*

Proof Let $A, B \in C(X)$ and $A \cap B = \emptyset$.

$$\left.\begin{matrix}(A, B \in C(X))(A \cap B = \emptyset) \\ f \text{ is a closed injection}\end{matrix}\right\} \Rightarrow (f[A], f[B] \in C(Y))(f[A] \cap f[B] = \emptyset) \left.\begin{matrix} \\ Y \text{ is } e^*\text{-normal}\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow (\exists G, H \in e^*O(Y))(f[A] \subseteq G)(f[B] \subseteq H)(G \cap H = \emptyset) \left.\begin{matrix} \\ f \text{ is } e^*\text{-}ge^*\text{-continuous}\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow (f^{-1}[G], f^{-1}[H] \in ge^*O(X))(A \subseteq f^{-1}[G])(B \subseteq f^{-1}[H])(f^{-1}[G \cap H] = \emptyset) \left.\begin{matrix} \\ (U_1 := e^*\text{-}int(f^{-1}[G]))(U_2 := e^*\text{-}int(f^{-1}[H]))\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow (U_1, U_2 \in e^*O(X))(A \subseteq U_1)(B \subseteq U_2)(U_1 \cap U_2 = \emptyset). \qquad \square$$

Corollary 2 *If $f : (X, \tau) \to (Y, \sigma)$ is a closed e^*-irresolute injection and Y is e^*-normal, then X is e^*-normal.*

Proof This is an immediate consequence since every e^*-irresolute function is e^*-ge^*-continuous. $\square$

Lemma 6 *If $f : (X, \tau) \to (Y, \sigma)$ is almost ge^*-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}[B]$, there exists a ge^*-open set V of Y such that $B \subseteq V$ and $f^{-1}[V] \subseteq U$.*

Proof *Necessity.* Let $B \subseteq Y,\ U \in RO(X)$ and $f^{-1}[B] \subseteq U$.

$$(U \in RO(X))(f^{-1}[B] \subseteq U) \Rightarrow (\backslash U \in RC(X))(\backslash U \subseteq \backslash f^{-1}[B] = f^{-1}[\backslash B]) \left.\begin{matrix} \\ f \text{ is almost } ge^*\text{-closed}\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow (f[\backslash U] \in ge^*C(Y))(f[\backslash U] \subseteq f[f^{-1}[\backslash B]] \subseteq \backslash B) \left.\begin{matrix} \\ V := \backslash f[\backslash U]\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow (V \in ge^*O(Y))(B \subseteq V)(f^{-1}[V] \subseteq U).$$

Sufficiency. Let $A \in RC(X),\ U \in O(Y)$ and $f[A] \subseteq U$.

$$(U \in O(Y))(A \in RC(X))(f[A] \subseteq U) \Rightarrow (\backslash U \subseteq Y)(\backslash A \in RO(X))(A \subseteq f^{-1}[U])$$

$$\Rightarrow (\backslash U \subseteq Y)(\backslash A \in RO(X))(f^{-1}[\backslash U] \subseteq \backslash A) \left.\begin{matrix} \\ \text{Hypothesis}\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow (\exists V \in ge^*O(Y))(\backslash U \subseteq V)(f^{-1}[V] \subseteq \backslash A)$$

$$\Rightarrow (\backslash V \in ge^*C(Y))(\backslash V \subseteq U)(A \subseteq f^{-1}[\backslash V])$$

$$\Rightarrow (e^*\text{-}cl(\backslash V) \subseteq U)(f[A] \subseteq \backslash V)$$

$$\Rightarrow e^*\text{-}cl(f[A]) \subseteq e^*\text{-}cl(\backslash V) \subseteq U. \qquad \square$$

Lemma 7 *If $f : (X, \tau) \to (Y, \sigma)$ is almost ge^*-closed, then for each closed set B of Y and each $U \in RO(X)$ containing $f^{-1}[B]$, there exists an e^*-open set V of Y such that $B \subseteq V$ and $f^{-1}[V] \subseteq U$.*

Proof Let $B \in C(Y),\ U \in RO(X)$ and $f^{-1}[B] \subseteq U$.

$$\left.\begin{matrix}(B \in C(Y))(f^{-1}[B] \subseteq U \in RO(X)) \\ f \text{ is almost } ge^*\text{-closed}\end{matrix}\right\} \Rightarrow (\exists W \in ge^*O(Y))(B \subseteq W)(f^{-1}[W] \subseteq\ U)$$

$$\Rightarrow (\backslash W \in ge^*C(Y))(\backslash W \subseteq \backslash B)(f^{-1}[W] \subseteq U)$$

$$\Rightarrow (\backslash W = e^*\text{-}cl(\backslash W) = \backslash e^*\text{-}int(W) \subseteq \backslash B)(f^{-1}[W] \subseteq U)$$

$$\Rightarrow (B \subseteq e^*\text{-}int(W))(f^{-1}[W] \subseteq U) \left.\begin{matrix} \\ V := e^*\text{-}int(W)\end{matrix}\right\} \Rightarrow (V \in e^*O(Y))(B \subseteq V)(f^{-1}[V] \subseteq U).$$

$\square$

Theorem 17 *If $f : (X, \tau) \to (Y, \sigma)$ is a continuous almost ge^*-closed surjection and X is normal, then Y is e^*-normal.*

Proof Let $A, B \in C(Y)$ and $A \cap B = \emptyset$.

$$\left.\begin{array}{r}(A, B \in C(Y))(A \cap B = \emptyset)\\ f \text{ is continuous}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (f^{-1}[A], f^{-1}[B] \in C(X))(f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B] = \emptyset) \left.\begin{array}{r}\\ X \text{ is normal}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U_1, U_2 \in O(X))(f^{-1}[A] \subseteq U_1)(f^{-1}[B] \subseteq U_2)(U_1 \cap U_2 = \emptyset) \left.\begin{array}{r}\\ (G := int(cl(U_1)))(H := int(cl(U_2)))\end{array}\right\} \Rightarrow$$

$$\Rightarrow (G, H \in RO(X))(f^{-1}[A] \subseteq U_1 \subseteq G)(f^{-1}[B] \subseteq U_2 \subseteq H)(G \cap H = \emptyset) \left.\begin{array}{r}\\ f \text{ is almost } ge^*\text{-closed}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists V_1, V_2 \in e^*O(Y))(A \subseteq V_1)(B \subseteq V_2)(f^{-1}[V_1] \subseteq G)(f^{-1}[V_2] \subseteq H)(G \cap H = \emptyset)$$

$$\Rightarrow (V_1, V_2 \in e^*O(Y))(A \subseteq V_1)(B \subseteq V_2)(V_1 \cap V_2 = \emptyset). \qquad \square$$

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On Generalizations of H-Space in Ideal Topological Spaces

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Abstract The main purpose of this chapter is to introduce and investigate a new class of spaces which contains the class of all H-spaces. Moreover, we offer another class independent to H-spaces but which contains the class of all Hausdorff spaces. Furthermore, we give not only various several properties and characterizations but also examples and counterexamples.

1 Introduction

Ideals in topological spaces have been considered since 1933. This topic has won its importance by the paper of [14]. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathscr{I}$ and $B \subseteq A$ implies $B \in \mathscr{I}$ and (ii) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$.

In an ideal topological space $(X, \tau, \mathscr{I})$, the local function $(\cdot)^*$ [8] is defined as $A^*(\mathscr{I}, \tau)$ (or simply A^*) $:= \{x \in X | (\forall U \in \mathscr{U}(x))(U \cap A \notin \mathscr{I})\}$, where $\mathscr{U}(x)$ is the collection of all open subsets containing $x \in X$. A Kuratowski closure operator cl^* for a topology $\tau^*(\mathscr{I}, \tau)$ called the $*$-topology, finer than τ is defined by $cl^*(A) = A \cup A^*$.

By the way, one of the most important abstract branch of mathematics is undoubtedly functional analysis. It is obvious that the fundamental concept of functional analysis is metric spaces. Metric spaces come across as an abstraction of the real line $\mathbb{R}$.

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The concept of metric space has been defined via the concept of distance. However, the concept of topological space is independent from the concept of distance except metric topological spaces. That is, topological spaces are the abstraction of metric spaces. Moreover, it is well known that every metric space is Hausdorff eventhough every topological space is not Hausdorff. Furthermore, a topological vector space is one of the fundamental structures studied in functional analysis. A topological vector space is a vector space that is also a topological space with the property that the vector space operations are also continuous functions. The concept of Hausdorff space is very important concept in topological vector space. Some spaces such as Banach spaces and Hilbert spaces are well-known examples of topological vector spaces which are Hausdorff.

The purpose of this paper is to introduce two new class of spaces called a Hausdorff space modulo $\mathscr{I}$ (resp. $\mathscr{I}$-H-space) which contains the class of all Hausdorff spaces (resp. H-spaces) and discuss some of its properties.

2 Preliminaries

Throughout this paper, by a space we will always mean a topological space. For a subset A of a space X, the closure and the interior of A will be denoted by $cl(A)$ and $int(A)$, respectively. An ideal topological space $(X, \tau, \mathscr{I})$ is called a Hayashi-Samuel space [2] if $\tau \cap \mathscr{I} = \{\emptyset\}$. A topological space (X, τ) is called an H-space [4] if for every $x \in X$, there is $y \in X$ such that $x \in G$, $y \in H$ and $G \cap H = \emptyset$ for some $G, H \in \tau$.

Definition 1 A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be semi-open [9] (resp. β-open [1], semi-$\mathscr{I}$-open [5], $\mathscr{I}$-open [7], quasi-$\mathscr{I}$-open [11]) if $A \subseteq cl(int(A))$ (resp. $A \subseteq cl(int(cl(A)))$, $A \subseteq cl^*(int(A))$, $A \subseteq int(A^*)$, $A \subseteq cl(int(A^*))$).

Definition 2 A space (X, τ) is said to be semi-Hausdorff [10] (resp. β-Hausdorff [11]) if for every two different points x and y of X, there exist disjoint semi-open (resp. β-open) sets U and V of X such that $x \in U$ and $y \in V$.

Definition 3 An ideal topological space $(X, \tau, \mathscr{I})$ is called an $\mathscr{I}$-Hausdorff [3] (resp. quasi-$\mathscr{I}$-Hausdorff [12], semi-$\mathscr{I}$-Hausdorff [6]) space if for every two different points x and y of X, there exist disjoint $\mathscr{I}$-open sets (resp. quasi-$\mathscr{I}$-open, semi-$\mathscr{I}$-open) U and V of X such that $x \in U$ and $y \in V$.

The family of all β-open (resp. semi-open, $\mathscr{I}$-open, semi-$\mathscr{I}$-open, quasi-$\mathscr{I}$-open) subsets of X is denoted by $\beta O(X, \tau)$ (resp. $SO(X, \tau)$, $\mathscr{I}O(X, \tau)$, $S\mathscr{I}O(X, \tau)$, $Q\mathscr{I}O(X, \tau)$).

3 $\mathscr{I}$-H-Spaces and Hausdorff Spaces Modulo $\mathscr{I}$

Definition 4 An ideal topological space $(X, \tau, \mathscr{I})$ is called an $\mathscr{I}$-H-space if for every $x \in X$, there is a point $y \in X$ and an open set B containing x such that $y \notin B^*$.

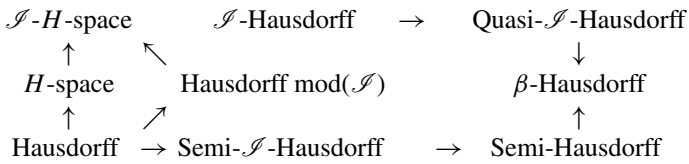
Definition 5 An ideal space $(X, \tau, \mathscr{I})$ is called a Hausdorff space modulo $\mathscr{I}$ (or Hausdorff mod($\mathscr{I}$)) if for each two distinct points $x, y \in X$, there is an open set B containing x such that $y \notin B^*$.

Theorem 1 *For an ideal topological space $(X, \tau, \mathscr{I})$, the following statements hold:*

(1) Every Hausdorff space is Hausdorff mod($\mathscr{I}$).
(2) Every H-space is an $\mathscr{I}$-H-space.
(3) If $(X, \tau, \mathscr{I})$ is Hausdorff mod($\mathscr{I}$), then it is an $\mathscr{I}$-H-space.

Proof (1) and (2) are obvious from $\emptyset \in \mathscr{I}$.
(3) Obvious. □

Remark 1 From Definitions 4, 5 and Theorem 1, we have the following diagram.



The following examples show that these implications are not reversible.

Example 1 Let $X := \{a, b, c\}$, $\tau := \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\mathscr{I} := \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then, $(X, \tau, \mathscr{I})$ is an $\mathscr{I}$-Hausdorff space but it is not an H-space.

Example 2 Let $X := \{a, b, c, d\}$, $\tau := \{\emptyset, X, \{a, b\}, \{c, d\}\}$ and $\mathscr{I} := \{\emptyset, \{a\}\}$. Then, $(X, \tau, \mathscr{I})$ is an H-space but it is not an $\mathscr{I}$-Hausdorff space.

Example 3 Let $X := \{a, b, c, d\}$, $\tau := \{\emptyset, X, \{b\}, \{b, d\}\}$ and $\mathscr{I} := 2^X$. Then, $(X, \tau, \mathscr{I})$ is Hausdorff mod($\mathscr{I}$) but it is not β-Hausdorff.

Example 4 Let $X := \{a, b, c\}$, $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathscr{I} := \{\emptyset\}$. Then, $(X, \tau, \mathscr{I})$ is semi-$\mathscr{I}$-Hausdorff but it is not $\mathscr{I}$-H-space.

For the other implications, the counterexamples are as shown in [6].

Remark 2 It is obvious to see that every ideal topological space $(X, \tau, \mathscr{I})$ is $\mathscr{I}$-Hausdorff when $\mathscr{I} = 2^X$ where 2^X is the collection of all subsets of X. In an ideal topological space $(X, \tau, \mathscr{I})$, it is clear that if $\mathscr{I} := \{\emptyset\}$, then Hausdorff space (resp. H-space) and $\mathscr{I}$-Hausdorff space (resp. and $\mathscr{I}$-H-space) coincide.

(1) : $\mathscr{I}$-H-space (2) : H-space (3) : Hausdorff
(4) : Hausdorff mod($\mathscr{I}$) (5) : $\mathscr{I}$-Hausdorff (6) : Semi-Hausdorff
(7) : Semi-$\mathscr{I}$-Hausdorff (8) : β-Hausdorff (9) : Quasi-$\mathscr{I}$-Hausdorff

There are several types of Hausdorf spaces in the literature such as given above. Now, we compare the relationships among introduced notion in this paper and studied in the literature.

Corollary 1 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $\mathscr{I} := \{\emptyset\}$, then we have the following diagram:*

$$\begin{array}{ccccccc} & & & & (5) & \to & (8) \equiv (9) \\ & & & & \uparrow & & \uparrow \\ (1) \equiv (2) & \leftarrow & (3) & \equiv & (4) & \to & (6) \equiv (7) \end{array}$$

Corollary 2 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $\mathscr{I} := 2^X$, then we have the following diagram:*

$$\begin{array}{ccccccccc} (3) \equiv (7) & \to & (6) & \to & (1) \equiv (4) & \leftarrow & (2) \\ & & & & \uparrow & & \\ (5) & \to & (9) & \to & (8) & & \end{array}$$

Theorem 2 *Let (X, τ) be a topological space and let $\mathscr{I}$ and $\mathscr{J}$ be two ideals on X such that $\mathscr{I} \subseteq \mathscr{J}$.*

(1) If $(X, \tau, \mathscr{I})$ is an $\mathscr{I}$-H-space, then $\left(X, \tau, \mathscr{J}\right)$ is a $\mathscr{J}$-H-space.
(2) If $(X, \tau, \mathscr{I})$ is Hausdorff mod($\mathscr{I}$), then $\left(X, \tau, \mathscr{J}\right)$ is Hausdorff mod($\mathscr{J}$).

Proof It is obvious from the fact that a topological space (X, τ) if $\mathscr{I} \subseteq \mathscr{J}$, then for all subsets A in X; $A^*(\mathscr{J}, \tau) \subseteq A^*(\mathscr{I}, \tau)$. □

Theorem 3 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, the following are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is an $\mathscr{I}$-H-space.
(2) For every point $x \in X$, there is a point $y \in X$ such that $x \in G$ and $y \in H$ and $G \cap H \in \mathscr{I}$ for some $G, H \in \tau$.
(3) For every point $x \in X$, there is a point $y \in X$ and an open subset H containing y such that $x \notin H^*$.

Proof (1) $\Rightarrow$ (2) : Let $(X, \tau, \mathscr{I})$ be an $\mathscr{I}$-H-space and $x \in X$.

$$\left.\begin{array}{r} x \in X \\ (X, \tau, \mathscr{I}) \text{ is an } \mathscr{I}\text{-}H\text{-space} \end{array}\right\} \Rightarrow (\exists y \in X)(\exists G \in \mathscr{U}(x))(y \notin G^*)$$

$\Rightarrow (\exists G \in \mathscr{U}(x))(\exists H \in \mathscr{U}(y))(G \cap H \in \mathscr{I})$.
(2) $\Rightarrow$ (3) : Obvious.
(3) $\Rightarrow$ (1) : Let $x \in X$.

$$\left.\begin{array}{r} x \in X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists y \in X)(\exists H \in \mathscr{U}(y))(x \notin H^*)$$

$\Rightarrow (\exists y \in X)(\exists G \in \mathscr{U}(x))(\exists H \in \mathscr{U}(y))(G \cap H \in \mathscr{I})$
$\Rightarrow (\exists y \in X)(\exists G \in \mathscr{U}(x))(y \notin G^*)$. □

Theorem 4 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, the following are equivalent:*

(1) $(X, \tau, \mathscr{I})$ is a Hausdorff space modulo $\mathscr{I}$.
(2) For each two different points $x, y \in X$, there exist an open subset G containing x and an open subset H containing y such that $G \cap H \in \mathscr{I}$.

Proof Obvious. □

Recall that if $\mathscr{I}$ is an ideal on a set X and A is a subset of X, then $\mathscr{I}_A := \{A \cap I | I \in \mathscr{I}\} = \{I \in \mathscr{I} | I \subseteq A\}$ is an ideal of A [13]. The following theorem shows that the property of Hausdorff modulo $\mathscr{I}$ is hereditary.

Theorem 5 *If $(X, \tau, \mathscr{I})$ is a Hausdorff space modulo $\mathscr{I}$ and $A \subseteq X$, then $(A, \tau_A, \mathscr{I}_A)$ is Hausdorff mod$(\mathscr{I}_A)$.*

Proof Let $A \subseteq X$ and let x and y be arbitrary two distinct points of A.

$$\left.\begin{matrix}(x, y \in A \subseteq X)(x \neq y) \\ \text{Hypothesis}\end{matrix}\right\} \Rightarrow (\exists G \in \mathscr{U}(x))(\exists H \in \mathscr{U}(y))(G \cap H \in \mathscr{I}) \left.\begin{matrix}\\ (G_A := A \cap G)(H_A := A \cap H)\end{matrix}\right\} \Rightarrow$$
$$\Rightarrow (x \in G_A \in \tau_A)(y \in H_A \in \tau_A)(G_A \cap H_A \in \mathscr{I}_A).$$ □

Remark 3 The following example shows that the property of an $\mathscr{I}$-H-space is not hereditary.

Example 5 Let $X := \{a, b, c\}$, $\tau := \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathscr{I} := \{\emptyset, \{a\}\}$. Let $A = \{b, c\}$. Then, $(X, \tau, \mathscr{I})$ is an $\mathscr{I}$-H-space but $(A, \tau_A, \mathscr{I}_A)$ is not an $\mathscr{I}$-H-space.

Theorem 6 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If for every $x \in X$ there is an open set G in $\mathscr{I}$ containing x, then $(X, \tau, \mathscr{I})$ is Hausdorff mod$(\mathscr{I})$.*

Proof Let x and y be two distinct points of X.

$$\left.\begin{matrix}x \in X \\ \text{Hypothesis}\end{matrix}\right\} \Rightarrow (\exists G \in \mathscr{U}(x))(G^* = \emptyset) \left.\begin{matrix}\\ x \neq y\end{matrix}\right\} \Rightarrow (G \in \mathscr{U}(x))(y \notin G^*).$$ □

Corollary 3 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If for every $x \in X$ there is an open set G in $\mathscr{I}$ containing x, then $(X, \tau, \mathscr{I})$ is an $\mathscr{I}$-H-space.*

Proof It is obvious from the fact that if $(X, \tau, \mathscr{I})$ is Hausdorff mod$(\mathscr{I})$, then it is an $\mathscr{I}$-H-space. □

Theorem 7 *A Hayashi-Samuel space $(X, \tau, \mathscr{I})$ is an $\mathscr{I}$-H-space if and only if (X, τ) is an H-space.*

Proof $(\Rightarrow)$: Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel $\mathscr{I}$-H-space and $x \in X$.

$$\left.\begin{matrix}x \in X \\ (X, \tau, \mathscr{I}) \text{ is an } \mathscr{I}\text{-}H\text{-space}\end{matrix}\right\} \Rightarrow (\exists y \in X)(\exists G \in \mathscr{U}(x))(y \notin G^*)$$
$$\Rightarrow (\exists y \in X)(\exists G \in \mathscr{U}(x))(\exists H \in \mathscr{U}(y))(G \cap H \in \mathscr{I}) \left.\begin{matrix}\\ (X, \tau, \mathscr{I}) \text{ is a Hayashi-Samuel space}\end{matrix}\right\} \Rightarrow$$
$$\Rightarrow (\exists y \in X)(\exists G \in \mathscr{U}(x))(\exists H \in \mathscr{U}(y))(G \cap H = \emptyset).$$

$(\Leftarrow)$: It is clear. □

Theorem 8 *A Hayashi-Samuel space $(X, \tau, \mathscr{I})$ is Hausdorff mod$(\mathscr{I})$ if and only if (X, τ) is a Hausdorff space.*

Proof $(\Rightarrow)$: Let $(X, \tau, \mathscr{I})$ be a Hayashi-Samuel Hausdorff space modulo $\mathscr{I}$ and $x, y \in X$ with $x \neq y$.

$$\left.\begin{array}{r}(x, y \in X)(x \neq y)\\ (X, \tau, \mathscr{I}) \text{ is Hausdorff mod}(\mathscr{I})\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists G \in \mathscr{U}(x))(\exists H \in \mathscr{U}(y))(G \cap H \in \mathscr{I})\\ (X, \tau, \mathscr{I}) \text{ is a Hayashi-Samuel space}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists G \in \mathscr{U}(x))(\exists H \in \mathscr{U}(y))(G \cap H = \emptyset).$$

$(\Leftarrow)$: It is clear. □

If $(X, \tau, \mathscr{I})$ is an ideal topological space, (Y, σ) is a topological space and $f : (X, \tau, \mathscr{I}) \to (Y, \sigma)$ is a function, then $f(\mathscr{I}) := \{f(I) | I \in \mathscr{I}\}$ is an ideal on Y [13]. The following theorems show that a Hausdorff space modulo $\mathscr{I}$ and an $\mathscr{I}$-H-space are a topological property.

Theorem 9 *If $(X, \tau, \mathscr{I})$ is Hausdorff mod$(\mathscr{I})$ and $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, f(\mathscr{I}))$ is a homeomorphism, then $(Y, \sigma, f(\mathscr{I}))$ is Hausdorff mod$(f(\mathscr{I}))$.*

Proof Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$.

$$\left.\begin{array}{r}(y_1, y_2 \in Y)(y_1 \neq y_2)\\ f \text{ is a homeomorphism}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists! x_1, x_2 \in X)(x_1 \neq x_2)(y_1 = f(x_1))(y_2 = f(x_2))\\ (X, \tau, \mathscr{I}) \text{ is Hausdorff mod}(\mathscr{I})\end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(\exists G \in \mathscr{U}(x_1))(\exists H \in \mathscr{U}(x_2))(G \cap H \in \mathscr{I})\\ f \text{ is a homeomorphism}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (f[G] \in \mathscr{U}(y_1))(f[H] \in \mathscr{U}(y_2))(f[G] \cap f[H] = f[G \cap H] \in f(\mathscr{I})).$$ □

Theorem 10 *If $(X, \tau, \mathscr{I})$ is an $\mathscr{I}$-H-space and $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, f(\mathscr{I}))$ is a homeomorphism, then $(Y, \sigma, f(\mathscr{I}))$ is a $f(\mathscr{I})$-H-space.*

Proof Let $y_1 \in Y$.

$$\left.\begin{array}{r}y_1 \in Y\\ f \text{ is a homeomorphism}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists! x_1 \in X)(y_1 = f(x_1))\\ (X, \tau, \mathscr{I}) \text{ is an } \mathscr{I}\text{-}H\text{-space}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists x_2 \in X)(\exists G \in \mathscr{U}(x_1))(x_2 \notin G^*)$$
$$\Rightarrow \left.\begin{array}{r}(\exists G \in \mathscr{U}(x_1))(\exists H \in \mathscr{U}(x_2))(G \cap H \in \mathscr{I})\\ f \text{ is a homeomorphism}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (y_2 = f(x_2))(f[G] \in \mathscr{U}(y_1))(f[H] \in \mathscr{U}(y_2))(f[G] \cap f[H] = f[G \cap H] \in f(\mathscr{I})).$$ □

If $(Y, \sigma, \mathscr{I})$ is an ideal topological space, (X, τ) is a topological space and $f : (X, \tau) \to (Y, \sigma, \mathscr{I})$ is a function, then $f^{-1}[\mathscr{I}] := \{f^{-1}[I] | I \in \mathscr{I}\}$ is an ideal on X.

Theorem 11 *Let $f : (X, \tau, f^{-1}[\mathscr{I}]) \to (Y, \sigma, \mathscr{I})$ be a non-constant continuous function. If $(Y, \sigma, \mathscr{I})$ is a Hausdorff space modulo $\mathscr{I}$, then $(X, \tau, f^{-1}[\mathscr{I}])$ is a $f^{-1}[\mathscr{I}]$-H-space.*

Proof Let $x \in X$.

$$\left.\begin{array}{r} x \in X \\ f \text{ is non-constant} \end{array}\right\} \Rightarrow \left.\begin{array}{c} (\exists y \in X)(f(x) \neq f(y)) \\ (Y, \sigma, \mathscr{I}) \text{ is Hausdorff mod}(\mathscr{I}) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists B \in \mathscr{U}(f(x)))(\exists H \in \mathscr{U}(f(y)))\left.\begin{array}{r} (B \cap H \in \mathscr{I}) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (f^{-1}[B] \in \mathscr{U}(x))(f^{-1}[H] \in \mathscr{U}(y))\left.\begin{array}{r} (f^{-1}[B] \cap f^{-1}[H] = f^{-1}[B \cap H] \in f^{-1}[\mathscr{I}]) \\ (U := f^{-1}[B])(V := f^{-1}[H]) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (y \in X)(U \in \mathscr{U}(x))(V \in \mathscr{U}(y))(U \cap V \in f^{-1}[\mathscr{I}]). \qquad \square$$

Theorem 12 *Let $f : (X, \tau, f^{-1}[\mathscr{I}]) \to (Y, \sigma, \mathscr{I})$ be a continuous injection. If $(Y, \sigma, \mathscr{I})$ is a Hausdorff space modulo $\mathscr{I}$, then $(X, \tau, f^{-1}[\mathscr{I}])$ is a Hausdorff space modulo $f^{-1}[\mathscr{I}]$.*

Proof Let $x, y \in X$ and $x \neq y$

$$\left.\begin{array}{r} (x, y \in X)(x \neq y) \\ f \text{ is injective} \end{array}\right\} \Rightarrow \left.\begin{array}{c} (f(x), f(y) \in Y)(f(x) \neq f(y)) \\ (Y, \sigma, \mathscr{I}) \text{ is Hausdorff mod}(\mathscr{I}) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists B \in \mathscr{U}(f(x)))(\exists H \in \mathscr{U}(f(y)))\left.\begin{array}{r} (B \cap H \in \mathscr{I}) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (f^{-1}[B] \in \mathscr{U}(x))(f^{-1}[H] \in \mathscr{U}(y))\left.\begin{array}{r} (f^{-1}[B] \cap f^{-1}[H] \in f^{-1}[\mathscr{I}]) \\ (U := f^{-1}[B])(V := f^{-1}[H]) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (U \in \mathscr{U}(x))(V \in \mathscr{U}(y))(U \cap V \in f^{-1}[\mathscr{I}]). \qquad \square$$

Corollary 4 *Let $f : (X, \tau, f^{-1}[\mathscr{I}]) \to (Y, \sigma, \mathscr{I})$ be a continuous injection. If $(Y, \sigma, \mathscr{I})$ is a Hausdorff space modulo $\mathscr{I}$, then $(X, \tau, f^{-1}[\mathscr{I}])$ is a $f^{-1}[\mathscr{I}]$-H-space.*

Proof It is obvious from the fact that if $(Y, \sigma, \mathscr{I})$ is Hausdorff mod$(\mathscr{I})$, then $(X, \tau, f^{-1}[\mathscr{I}])$ is a Hausdorff space modulo $f^{-1}[\mathscr{I}]$ and so it is a $f^{-1}[\mathscr{I}]$-H-space. $\square$

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A Novel Mappings in Supra Topological Space with Neutrosophic Environment

Mani Parimala, Muthusamy Karthika, and Cenap Ozel

Abstract Neutrosophic continuous map and other maps, such as closed and open, are studies of the relationship between two neutrosohic supra topological spaces. In the same way, we incorporate the notion neutrosophic $\alpha\psi$-supra open set into the neutrosophic continuous map and other maps and defined new functions in neutrosophic supra topological space, and their properties are studied in this chapter. Some examples are provided to support the defined notion.

1 Introduction

The application of topological spaces can be seen in various fields like Physics, Computer Science, Biology, Differential Equations, and Functional Analysis. In particular, functional analysis is to investigate linear space with algebraic and topological structure. Functional analysis is a branch of analysis that investigates the properties of mapping of classes of function from one topological vector space to another. Epsilon-delta version is used in functional analysis and the same idea is used in topology that is the preimage of each open set is open. Elements of topological spaces are called the open sets. Zadeh [14] established the concept to address the uncertainty of an element in the set whose grade lies between 0 and 1 and such a set is called a fuzzy set. Coker [2] investigated the topological space under a fuzzy set. It gains the attention of the topological researchers.

Zadeh's theory fails to address the non-belonging grade of an element in the set, so Atanassov [1] addressed the both belonging and non-belonging grades of an element

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in the set. Later, Coker [2] coined the idea of topological space under intuitionistic fuzzy set. Many weak and strong forms of open and closed sets, continuity, connectedness, and compactness on Coker's notion were introduced and studied by many researchers. Smarandache [13] found that the element also has a grade of indeterminacy, including the grade of belonging and non-belonging and such a triplet set is called a neutrosophic set(NS). Salaman [11] developed the concept of novel topological space called a neutrosophic topological space if it contains $\tilde{0}$, $\tilde{1}$, closed under arbitrary union and finite intersection. Many weak and strong forms of open and closed sets [3, 5, 8, 9, 11] were introduced and studied. Also, various continuous functions, connectedness, and compactness [4, 6, 7] were established and studied by various authors.

Supra topological space with neutrosophic information is a topological space whose collection contains only $\tilde{0}$, $\tilde{1}$ and closed under arbitrary union. Parimala et al. introduced the weak form of open sets namely, semi-supra open set with neutrosophic information [3] and its continuous functions, $\alpha\psi$-supra open sets with neutrosophic information [9].

A map from topological space (X, κ) to topological space (X', κ') is said to be continuous map if the pre-image is open for all open sets in κ'. The notion of continuous map motivates to study the neutrosophic $\alpha\psi$-supra continuous map(briefly,$N_{\alpha\psi}$SCM) and other maps such as neutrosophic $\alpha\psi$-supra open map(briefly,$N_{\alpha\psi}$SOM), neutrosophic $\alpha\psi$-supra closed map(briefly,$N_{\alpha\psi}$SCLM). Neutrosophic $\alpha\psi$-supra homeomorphism is introduced to investigate the bijection and continuity of the map and continuity of its inverse map.

The structure of the paper is given below: Sect. 2, contains the definition of $N_{\alpha\psi}$SCM and neutrosophic $\alpha\psi$-supra irresolute (briefly,$N_{\alpha\psi}$SIM) and its properties. $N_{\alpha\psi}$SOM and $N_{\alpha\psi}$SCLM defined and studied in Sect. 3. Sections 4 and 5, carry the definition of neutrosophic $\alpha\psi$-supra homeomorphism(briefly,$N_{\alpha\psi}$SHOM), its properties and conclusion, respectively.

2 Neutrosophic Supra Topological Space

NS is a tool which can be used in topological space, operations research, probability theory, etc.

2.1 Preliminaries

The definition of NS, neutrosophic supra topological space (briefly, NSTpS), neutrosophic $\alpha\psi$ closed set(briefly, $N_{\alpha\psi}SCS$), and some operations are presented in this section.

Definition 1 ([13, 14]) A NS $\mathcal{N}$ on a $X \neq \emptyset$ is defined by $\mathcal{N} = \{(x, \mathcal{G}_N(x), \mathcal{I}_N(x), \mathcal{H}_N(x)) : x \in X\}$ with the condition that the addition of belonging ($\mathcal{G}_N : X \to [0, 1]$), indeterminacy ($\mathcal{I}_N : X \to [0, 1]$) and non-belonging grades ($\mathcal{H}_N : X \to [0, 1]$) lies between 0 and 3.

Definition 2 ([14]) Let $\mathcal{N}_1 = \{\langle \mathfrak{a}, \mathcal{G}_{\mathcal{N}_1}(\mathfrak{a}), \mathcal{I}_{\mathcal{N}_1}(\mathfrak{a}), \mathcal{H}_{\mathcal{N}_1}(\mathfrak{a})\rangle : \mathfrak{a} \in X\}$ and $\mathcal{N}_2 = \{\langle \mathfrak{a}, \mathcal{G}_{\mathcal{N}_2}(\mathfrak{a}), \mathcal{I}_{\mathcal{N}_2}(\mathfrak{a}), \mathcal{H}_{\mathcal{N}_2}(\mathfrak{a})\rangle : \mathfrak{a} \in X\}$ be NSs. Then the subset, complement, intersection, and union of NSs are defined below:

(i) $\mathcal{N}_1 \subseteq \mathcal{N}_2$ iff $\mathcal{G}_{\mathcal{N}_1}(\mathfrak{a}) \leq \mathcal{G}_{\mathcal{N}_2}((\mathfrak{a}))$, $\mathcal{I}_{\mathcal{N}_1}(\mathfrak{a}) \geq \mathcal{I}_{\mathcal{N}_2}(\mathfrak{a})$ and $\mathcal{H}_{\mathcal{N}_1}(\mathfrak{a}) \geq \mathcal{H}_{\mathcal{N}_2}(\mathfrak{a})$;
(ii) ${\mathcal{N}_1}^C$ is $\{\langle \mathfrak{a}, \mathcal{H}_{\mathcal{N}_1}(\mathfrak{a}), 1 - \mathcal{I}_{S_1}(\mathfrak{a}), \mathcal{G}_{\mathcal{N}_1}(\mathfrak{a})\rangle : \mathfrak{a} \in X\}$;
(iii) $\mathcal{N}_1 \cap \mathcal{N}_2$ is $\{\langle \mathfrak{a}, \mathcal{G}_{\mathcal{N}_1}(\mathfrak{a}) \wedge \mathcal{G}_{\mathcal{N}_2}(\mathfrak{a}), \mathcal{I}_{\mathcal{N}_1}(\mathfrak{a}) \vee \mathcal{I}_{\mathcal{N}_2}(\mathfrak{a}), \mathcal{H}_{\mathcal{N}_1}(\mathfrak{a}) \vee \mathcal{H}_{\mathcal{N}_2}(\mathfrak{a})\rangle : \mathfrak{a} \in X\}$;
(iv) Union of $\mathcal{N}_1$ and $\mathcal{N}_2$ is $\{\langle \mathfrak{a}, \mathcal{G}_{\mathcal{N}_1}(\mathfrak{a}) \vee \mathcal{G}_{\mathcal{N}_2}(\mathfrak{a}), \mathcal{I}_{\mathcal{N}_1}(\mathfrak{a}) \wedge \mathcal{I}_{\mathcal{N}_2}(\mathfrak{a}), \mathcal{H}_{\mathcal{N}_1}(\mathfrak{a}) \wedge \mathcal{H}_{\mathcal{N}_2}(\mathfrak{a})\rangle : \mathfrak{a} \in X\}$.

Definition 3 ([3]) A subcollection κ of NS on $X \neq \emptyset$ is said to be a NSTpS on X if

(i) $\tilde{0}, \tilde{1} \in \tau$
(ii) $\bigcup_{j=i}^{\infty} \mathcal{N}_j \in \tau,\ for\ \{\mathcal{N}_j\}_{j=1}^{\infty} \in \tau$.

Definition 4 ([10]) If $\mathcal{U} \subseteq N\psi Sint(\mathcal{N})$ whenever $\mathcal{U} \subseteq \mathcal{N}$ and $\mathcal{U}$ is a neutrosophic α-supra closed set. Then the NS $\mathcal{N}$ is called a neutrosophic $\alpha\psi$-supra open set(briefly, $N_{\alpha\psi}SOS$). The complement of a $N_{\alpha\psi}SOS$ is called a $N_{\alpha\psi}SCS$.

3 $\alpha\psi$-Supra Continuous Map Under Neutrosophic Environment

The definition of $N_{\alpha\psi}$SCM, $N_{\alpha\psi}$SIM, and its properties are presented in this section.

Definition 5 A map Λ from NSTpS (X, κ) to NSTpS (X', κ') is called a $N_{\alpha\psi}$SCM if $\Lambda^{-1}(\mathcal{B})$ is a $N_{\alpha\psi}$SOS in (X, κ) for each neutrosophic supra open set(in short, NSOS) $\mathcal{B}$ in (X', κ').

Example 1 Let $X = \{x, y\}$ and $Y = \{a, b\}$ be a non-empty sets. Let $\mathcal{A}$ and $\mathcal{B}$ be NSs of X and X', respectively, defined as
$\mathcal{A}(x) = (0.3, 0.4, 0.4)$, $\mathcal{A}(y) = (0.3, 0.4, 0.4)$, and $\mathcal{B}(x) = (0.3, 0.4, 0.4)$, $\mathcal{B}(y) = (0, 1, 1)$. Define $\kappa = \{\tilde{0}, \tilde{1}, \mathcal{A}\}$, and $\kappa' = \{\tilde{0}, \tilde{1}, \mathcal{B}\}$. Clearly, κ and κ' are NSTpS on X and X'. Consider the map $\Lambda : (X, \kappa) \to (X', \kappa')$ defined by $\Lambda(x) = a$, $\Lambda(y) = b$. Then $\Lambda^{-1}(\tilde{0}) = \tilde{0}$, $\Lambda^{-1}(\tilde{1}) = \tilde{1}$ and $\Lambda^{-1}(\mathcal{A}) = \mathcal{B}$ are $N_{\alpha\psi}SOS$ of (X, κ). Hence, Λ is continuous.

Theorem 1 *Let Λ be a map from NSTpS (X, κ) to NSTpS (X', κ'). Then the following holds:*

i. Λ *is* $N_{\alpha\psi}SCM$,
ii. $\Lambda(N_{\alpha\psi}SCl(\mathscr{A})) \subseteq N_{\alpha\psi}SCl(\Lambda(\mathscr{A}))$ *for each NS* $\mathscr{A}$ *in X,*
iii. $N_{\alpha\psi}SCl(\Lambda^{-1}(\mathscr{B})) \subseteq \Lambda^{-1}(N_{\alpha\psi}SCl(\mathscr{B}))$ *for each NS* $\mathscr{B}$ *in* X'.
iv. $\Lambda^{-1}(N_{\alpha\psi}Sint(\mathscr{B})) \subseteq N_{\alpha\psi}Sint(\Lambda^{-1}(\mathscr{B}))$ *for each NS* $\mathscr{B}$ *in* X'

Proof $(i) \to (ii)$. Let Λ be a $N_{\alpha\psi}SCM$ and $\Lambda^{-1}(\mathscr{B}) = \mathscr{A}$ is a $N_{\alpha\psi}SCS$ of X for each NSCS $\mathscr{B}$ of X'. If $\mathscr{B}$ of $\Lambda(\mathscr{A})$ is a $N_{\alpha\psi}SCS$ of X'. Then

$$\Lambda(\mathscr{A}) \subseteq N_{\alpha\psi}SCl(\Lambda(\mathscr{A}))$$

is a $N_{\alpha\psi}SCS$ of X', $\Lambda^{-1}(N_{\alpha\psi}SCl(\Lambda(\mathscr{A}))$ is a $N_{\alpha\psi}SCS$ of X.
$N_{\alpha\psi}SCl(\mathscr{A}) \subseteq N_{\alpha\psi}SCl(\Lambda^{-1}(\Lambda(\mathscr{A})) \subseteq N_{\alpha\psi}SCl(\Lambda^{-1}(N_{\alpha\psi}SCl(\Lambda(\mathscr{A}))) =$
$\Lambda^{-1}(N_{\alpha\psi}SCl(\Lambda(\mathscr{A}))$. Thus $\Lambda(N_{\alpha\psi}SCl(\mathscr{A})) \subseteq N_{\alpha\psi}SCl(\Lambda(\mathscr{A})$.
$(ii) \to (iii)$. Let $\mathscr{A} = \Lambda^{-1}(\mathscr{B})$. Then from (ii)
$\Lambda(N_{\alpha\psi}SCl(\Lambda^{-1}(\mathscr{B}))) \subseteq N_{\alpha\psi}SCl(\Lambda(\Lambda^{-1}(\mathscr{B}))) = N_{\alpha\psi}SCl(\mathscr{B})$.
Hence $N_{\alpha\psi}SCl(\Lambda^{-1}(\mathscr{B})) \subseteq \Lambda^{-1}(N_{\alpha\psi}SCl(\mathscr{B}))$ for each $\mathscr{B}$ of X'.
$(iii) \to (iv)$. Let $\mathscr{B}$ be a $N_{\alpha\psi}SCS$ in X'. Then $X'\backslash\mathscr{B} = \mathscr{B}^c$ is a $N_{\alpha\psi}SOS$ in X' and $\Lambda^{-1}(X'\backslash\mathscr{B}) = \Lambda^{-1}(X)\backslash\Lambda^{-1}(\mathscr{B}) = X\backslash\Lambda^{-1}(\mathscr{B})$ is a $N_{\alpha\psi}SOS$ in X. By (iii), $N_{\alpha\psi}SCl(\Lambda^{-1}(\mathscr{B})) \subseteq \Lambda^{-1}(N_{\alpha\psi}SCl(\mathscr{B}))$. Taking complement on both the sides, we get

$$[\Lambda^{-1}(N_{\alpha\psi}SCl(\mathscr{B}))]^c \subseteq [N_{\alpha\psi}SCl(\Lambda^{-1}(\mathscr{B}))]^c.$$

Thus $\Lambda^{-1}(N_{\alpha\psi}Sint(\mathscr{B}^c)) \subseteq N_{\alpha\psi}Sint(\Lambda^{-1}(\mathscr{B}^c))$ for each NS $\mathscr{B}$ in X'.

Definition 6 A map Λ from NSTpS (X, κ) to NSTpS (X', κ') is called a $N_{\alpha\psi}$SIM if for each $N_{\alpha\psi}$SOS $\mathscr{B}$ in X', there is a $N_{\alpha\psi}$SOS $\Lambda^{-1}(\mathscr{B})$ in X.

Theorem 2 *Let $(X, \kappa), (X', \kappa'), (X'', \kappa'')$ be three NSTpS and the map Λ from NSTpS (X, κ) to NSTpS (X', κ') and the map Ω from NSTpS (X', κ') to NSTpS (X'', κ'') be $N_{\alpha\psi}SCM$. Then $\Omega \circ \Lambda : (X, \kappa) \to (X'', \kappa'')$ is a $N_{\alpha\psi}SIM$.*

Proof By hypothesis, $\Lambda^{-1}(V)$ is $N_{\alpha\psi}SOS$ in X, for all NSOS V in X' and $\Omega^{-1}(W)$ is $N_{\alpha\psi}SOS$ in X', for all NSOS W in X''. To prove $(\Omega \circ \Lambda)^{-1}(W)$ is $N_{\alpha\psi}SOS$ in X. Let $\Omega^{-1}(W) = V$. We know that $(\Omega \circ \Lambda)^{-1}(W) = \Lambda^{-1}(\Omega^{-1}(W))$. Then $\Lambda^{-1}(\Omega^{-1}(W))$ is a $N_{\alpha\psi}SOS$ in X, for all $N_{\alpha\psi}SOS$ $\Omega^{-1}(W)$ in X''. Hence, $\Omega \circ \Lambda : (X, \kappa) \to (X'', \kappa'')$ is a $N_{\alpha\psi}SIM$.

Theorem 3 *If Ω is a Neutrosophic supra irresolute map from NSTpS (X, κ) to NSTpS (X', κ') then Ω is a $N_{\alpha\psi}SIM$.*

Proof By hypothesis, Ω is a Neutrosophic supra irresolute map from NSTpS (X, κ) to NSTpS (X', κ'). Then $\Omega^{-1}(\mathscr{U})$ is a NSOS in X, $\forall$ NSOS $\mathscr{U}$ in X'. We know that every NSOS is a $N_{\alpha\psi}SOS$. Therefore, $\Omega^{-1}(\mathscr{U})$ is a $N_{\alpha\psi}SOS$ in X for all $N_{\alpha\psi}SOS\mathscr{U}$ in X'. Hence, the map Ω is a $N_{\alpha\psi}SIM$.

Theorem 4 *If Λ is a $N_{\alpha\psi}SIM$ from NSTpS (X, κ) to NSTpS (X', κ') and Ω is a $N_{\alpha\psi}SCM$ from NSTpS (X', κ') to NSTpS (X'', κ''). Then $\Omega \circ \Lambda$ is a $N_{\alpha\psi}SIM$.*

Proof Given: $\Lambda^{-1}(\mathscr{U})$ is a $N_{\alpha\psi}SOS$ in X for all $N_{\alpha\psi}SOS\mathscr{U}$ in X' and $\Omega^{-1}(W)$ is $N_{\alpha\psi}SOS$ in X', for all NSOS W in X''. To prove $(\Omega \circ \Lambda)^{-1}(W)$ is $N_{\alpha\psi}SOS$ in X. Let $\Omega^{-1}(W) = \mathscr{U}$. We know that $(\Omega \circ \Lambda)^{-1}(W) = \Lambda^{-1}(\Omega^{-1}(W))$. Then $\Lambda^{-1}(\Omega^{-1}(W))$ is a $N_{\alpha\psi}SOS$ in X, for all $N_{\alpha\psi}SOS$ $\Omega^{-1}(W)$ in X''. Hence, $\Omega \circ \Lambda : (X, \kappa) \to (X'', \kappa'')$ is a $N_{\alpha\psi}SIM$.

4 On $\alpha\psi$-supra Open and $\alpha\psi$-supra Closed Map Under Neutrosophic Environment

The definition of neutrosophic $\alpha\psi$-supra open map (briefly, $N_{\alpha\psi}SOM$) and neutrosophic $\alpha\psi$-supra closed map (briefly, $N_{\alpha\psi}SCLM$) and its properties are presented in this section.

Definition 7 A map Λ from NSTpS (X, κ) to NSTpS (X', κ') is called a $N_{\alpha\psi}SOM$, if $\Lambda(\mathscr{A})$ is a $N_{\alpha\psi}SOS$ in X' for all NSOS $\mathscr{A}$ in X.

Theorem 5 *If a map Λ from NSTpS (X, κ) to NSTpS (X', κ') is neutrosophic supra open map, then it is a $N_{\alpha\psi}SOM$.*

Proof Given $\Lambda(\mathscr{A})$ is a NSOS in X' for all NSOS $\mathscr{A}$ in X. We know that every NSOS is a $N_{\alpha\psi}SOS$. Therefore, $\Lambda(\mathscr{A})$ is a $N_{\alpha\psi}SOS$ in X'. Hence Λ is a $N_{\alpha\psi}SOM$. But if a map Λ is a $N_{\alpha\psi}SOM$, then Λ is not a neutrosophic supra open map. Since every $N_{\alpha\psi}SOS$ may not be a NSOS.

Theorem 6 *If a map Λ from NSTpS (X, κ) to NSTpS (X', κ') is neutrosophic supra open map and Ω from NSTpS (X', κ') to NSTpS (X'', κ'') is a $N_{\alpha\psi}SOM$. Then $\Omega \circ \Lambda$ is a $N_{\alpha\psi}SOM$.*

Theorem 7 *Let Λ be a map from NSTpS (X, κ) to NSTpS (X', κ'). Then the following are equivalent:*

i. *Λ is a $N_{\alpha\psi}SOM$,*
ii. *$\Lambda(N_{\alpha\psi}Sint(\mathscr{A})) \subseteq N_{\alpha\psi}Sint(\Lambda(\mathscr{A}))$ for each NS $\mathscr{A}$ of X,*
iii. *$N_{\alpha\psi}Sint(\Lambda^{-1}(\mathscr{B})) \subseteq \Lambda^{-1}(N_{\alpha\psi}Sint(\mathscr{B}))$ for each NS $\mathscr{B}$ of X'.*

Proof $(i) \to (ii)$. Let $\mathscr{A}$ be any NS of X. We know that $N_{\alpha\psi}Sint(\mathscr{A})$ is a $N_{\alpha\psi}SOS$ in X. Given Λ is a $N_{\alpha\psi}SOM$. $\Lambda(N_{\alpha\psi}Sint(\mathscr{A}))$ is a $N_{\alpha\psi}SOS$ in X'.

$$\Lambda(N_{\alpha\psi}Sint(\mathscr{A})) = N_{\alpha\psi}Sint(\Lambda(N_{\alpha\psi}Sint(\mathscr{A}))) \subseteq N_{\alpha\psi}Sint(\Lambda(\mathscr{A}))$$

.

$(ii) \to (iii)$. Let $\mathscr{B}$ be any Ns of X'. By (ii), $\Lambda(N_{\alpha\psi}Sint(\mathscr{A})) \subseteq N_{\alpha\psi}Sint(\Lambda(\mathscr{A}))$ $\Longrightarrow$ $\Lambda(N_{\alpha\psi}Sint(\Lambda^{-1}(\mathscr{B}))) \subseteq N_{\alpha\psi}Sint(\Lambda(\Lambda^{-1}(\mathscr{B}))) = N_{\alpha\psi}Sint(\mathscr{B})$. Thus, $N_{\alpha\psi}Sint(\Lambda^{-1}(\mathscr{B})) \subseteq \Lambda^{-1}(N_{\alpha\psi}Sint(\mathscr{B}))$.
$(iii) \to (i)$. Let $\mathscr{A}$ be any NSOS in X. Then $N_{\alpha\psi}Sint(\mathscr{A}) = \mathscr{A}$ and $\Lambda(\mathscr{A}) = \mathscr{B}$ is a NS of Y. Given $N_{\alpha\psi}Sint(\Lambda^{-1}(\mathscr{B})) \subseteq \Lambda^{-1}(N_{\alpha\psi}Sint(\mathscr{B}))$. Then

$$\mathscr{A} \subseteq \Lambda^{-1}(N_{\alpha\psi}Sint(\Lambda(\mathscr{A})))$$

$$\Lambda(\mathscr{A}) \subseteq N_{\alpha\psi}Sint(\Lambda(\mathscr{A})) \longrightarrow (1)$$

Let $\mathscr{A}$ be a NS of X, then $N_{\alpha\psi}Sint(\mathscr{A}) \subseteq \mathscr{A}$ and $\Lambda(\mathscr{A})$ is a NS of X'.

$$\Lambda(N_{\alpha\psi}Sint(\mathscr{A})) \subseteq \Lambda(\mathscr{A}) \longrightarrow (2).$$

From (1) and (2), $\Lambda(\mathscr{A}) = \Lambda(N_{\alpha\psi}Sint(\mathscr{A}))$. Therefore, $\Lambda(\mathscr{A})$ is a $N_{\alpha\psi}SOS$ in X'. Hence the map Λ is a $N_{\alpha\psi}SOM$.

Definition 8 A map Λ from NSTpS (X, κ) to NSTpS (X', κ') is called a $N_{\alpha\psi}SCLM$, if $\Lambda(\mathscr{A})$ is a $N_{\alpha\psi}SCS$ in X' for each NSCS $\mathscr{A}$ in X.

Theorem 8 *If a map Λ from NSTpS (X, κ) to NSTpS (X', κ') is neutrosophic supra closed map(briefly, NSCM), then it is a $N_{\alpha\psi}SCLM$.*

Proof Let $\mathscr{A}$ be a NSCS in (X, κ). Since Λ is a NSCM. Then $\Lambda(\mathscr{A})$ is a NSCS in (X', κ'). Every neutrosophic supra closed set is a $N_{\alpha\psi}SCS$. Therefore, $\Lambda(\mathscr{A})$ is a $N_{\alpha\psi}SCS$ in (X', κ'). Hence Λ is a $N_{\alpha\psi}SCLM$.

Theorem 9 *If a map Λ from NSTpS (X, κ) to NSTpS (X', κ') is neutrosophic supra closed map and Ω from NSTpS (X', κ') to NSTpS (X'', κ'') is a $N_{\alpha\psi}SCLM$. Then $\Omega \circ \Lambda$ is a $N_{\alpha\psi}SCLM$.*

Proof Let $\mathscr{A}$ be a NSCS in (X, κ). Then $\Lambda(\mathscr{A})$ is a NSCS in (X', κ'). $(\Omega \circ \Lambda)(\mathscr{A}) = \Omega(\Lambda(\mathscr{A}))$ is a $N_{\alpha\psi}SCS$ in (X'', κ''). Since Ω is a $N_{\alpha\psi}SCLM$. Hence $\Omega \circ \Lambda$ is a $N_{\alpha\psi}SCLM$.

Theorem 10 *Let Λ be a map from NSTpS (X, κ) to NSTpS (X', κ'). Then the following are equivalent:*

i. *Λ is a $N_{\alpha\psi}SCLM$,*
ii. *$N_{\alpha\psi}Scl(\Lambda(\mathscr{A})) \subseteq \Lambda(N_{\alpha\psi}Scl(\mathscr{A})$ for each NS $\mathscr{A}$ of X,*
iii. *$\Lambda^{-1}(N_{\alpha\psi}Scl(\mathscr{B})) \subseteq N_{\alpha\psi}Scl(\Lambda^{-1}(\mathscr{B}))$ for each NS $\mathscr{B}$ of X'.*

Proof $(i) \to (ii)$. Let $\mathscr{A}$ be any NS of X. Let $\Lambda : (X, \kappa) \to (X', \kappa')$ be a $N_{\alpha\psi}SCLM$, then $\Lambda(\mathscr{A})$ is $N_{\alpha\psi}SCS$ in X' for each NSCS $\mathscr{A}$ in X. Since $\Lambda(\mathscr{A})$ is $N_{\alpha\psi}SCS$ in X', $\Lambda(\mathscr{A}) \subseteq \Lambda(N_{\alpha\psi}(\mathscr{A}))$, then $N_{\alpha\psi}(\Lambda(\mathscr{A})) \subseteq N_{\alpha\psi}(\Lambda(N_{\alpha\psi}(\mathscr{A}))) = \Lambda(N_{\alpha\psi}(\mathscr{A}))$. Thus $N_{\alpha\psi}(\Lambda(\mathscr{A})) \subseteq \Lambda(N_{\alpha\psi}(\mathscr{A}))$.

$(ii) \rightarrow (iii)$. Let $\Lambda(\mathscr{A}) = \mathscr{B}$ be any NS of X'. Clearly $\Lambda^{-1}(\mathscr{B})$ is a NS of X. By (ii), $N_{\alpha\psi}Scl(\mathscr{B}) \subseteq \Lambda(N_{\alpha\psi}(\Lambda^{-1}(\mathscr{B})))$, then $\Lambda^{-1}(N_{\alpha\psi}Scl(\mathscr{B})) \subseteq N_{\alpha\psi}(\Lambda^{-1}(\mathscr{B}))$.
$(iii) \rightarrow (ii)$. Let $\mathscr{A}$ be any NS of X, then $\Lambda(\mathscr{A})$ is a NS of X'. Since Λ is one-to-one,

$$\Lambda^{-1}(N_{\alpha\psi}Scl(\mathscr{B})) \subseteq N_{\alpha\psi}Scl(\Lambda^{-1}(\mathscr{B})) = N_{\alpha\psi}Scl(\mathscr{A}),$$

$\Lambda^{-1}(N_{\alpha\psi}Scl(\Lambda(\mathscr{A}))) \subseteq N_{\alpha\psi}Scl(\mathscr{A})$. Thus $N_{\alpha\psi}Scl(\Lambda(\mathscr{A})) \subseteq \Lambda(N_{\alpha\psi}Scl(\mathscr{A}))$.
$(ii) \rightarrow (i)$. Let $\mathscr{A}$ be NSCS of X, then $\mathscr{A} = NScl(\mathscr{A})$. By (ii), Every NSCS is a $N_{\alpha\psi}SCS$. Therefore, $\mathscr{A} = N_{\alpha\psi}Scl(\mathscr{A})$.

$$N_{\alpha\psi}Scl(\Lambda(\mathscr{A})) \subseteq \Lambda(N_{\alpha\psi}Scl(\mathscr{A})) = \Lambda(\mathscr{A})$$

$N_{\alpha\psi}Scl(\Lambda(\mathscr{A})) \subseteq \Lambda(\mathscr{A}) \longrightarrow (1)$.
If $\Lambda(\mathscr{A})$ is a NS of X' then $\Lambda(\mathscr{A}) \subseteq N_{\alpha\psi}Scl(\Lambda(\mathscr{A})) \longrightarrow (2)$.
From (1) and (2), $\Lambda(\mathscr{A}) = N_{\alpha\psi}Scl(\Lambda(\mathscr{A})) \longrightarrow (2)$. Hence $\Lambda(\mathscr{A})$ is a $N_{\alpha\psi}SCS$ in X'.

5 Neutrosophic $\alpha\psi$-supra Homeomorphism

We have introduced the concept of neutrosophic supra homeomorphism(briefly, NSHOM), neutrosophic $\alpha\psi$ supra homeomorphism(briefly, $N_{\alpha}\psi SHOM$) and its properties with few examples in this section.

Definition 9 A map Λ from NSTpS (X, κ) to NSTpS (X', κ') is called NSHOM if Λ is bijective neutrosophic supra continuous and Λ^{-1} is a neutrosophic supra continuous.

Example 2 Let $X = \{x, y, z\}$ be non-empty set and $\mathscr{A}$ be a NS of X. The NS $\mathscr{A}$ is defined as $\mathscr{A}(x) = (0.3, 0.3, 0.4)$, $\mathscr{A}(y) = (0.4, 0.4, 0.2)$. Define $\kappa = \{\tilde{0}, \tilde{1}, \mathscr{A}\}$ and clearly it is a NSTpS. The identity map $\Lambda : (X, \kappa) \rightarrow (X, \kappa)$ is a NSHOM. Since Λ is a bijective neutrosophic supra continuous and Λ^{-1} also neutrosophic supra continuous.

Definition 10 A map Λ from NSTpS (X, κ) to NSTpS (X', κ') is called neutrosophic ψ-supra homeomorphism(briefly, $N_{\psi}SHOM$) if Λ is bijective neutrosophic ψ-supra continuous and Λ^{-1} is a neutrosophic ψ-supra continuous.

Definition 11 A map Λ from NSTpS (X, κ) to NSTpS (X', κ') is called neutrosophic α-supra homeomorphism(briefly, $N_{\alpha}SHOM$) if Λ is bijective neutrosophic α-supra continuous and Λ^{-1} is a neutrosophic α-supra continuous.

Definition 12 A map Λ from NSTpS (X, κ) to NSTpS (X', κ') is called neutrosophic $\alpha\psi$-supra homeomorphism(briefly, $N_{\alpha\psi}SHOM$) if Λ is bijective neutrosophic $\alpha\psi$-supra continuous and Λ^{-1} is a neutrosophic $\alpha\psi$-supra continuous.

Theorem 11 *Every neutrosophic supra homeomorphism is a $N_{\alpha\psi}SHOM$.*

Proof Let the map Λ be NSHOM. Then Λ is bijective, Λ and Λ^{-1} are neutrosophic supra continuous. We know that every neutrosophic supra continuous is a $N_{\alpha\psi}SCM$. Therefore, Λ and Λ^{-1} are $N_{\alpha\psi}SCM$. Hence Λ is a $N_{\alpha\psi}SHOM$.

Example 3 Let $X = \{x, y, z\}$ and $Y = \{a, b, c\}$ be non-empty set. Let $\mathscr{A}$ and $\mathscr{B}$ be NS of X and Y respectively. $\mathscr{A}(x) = (0.5, 03, 0.2)$, $\mathscr{A}(y) = (0.6, 0.2, 0.2)$, $\mathscr{A}(z) = (0.4, 0.4, 0.2)$, $\mathscr{B}(a) = (0.4, 0.4, 0.2)$, $\mathscr{B}(b) = (0.5, 0.3, 0.2)$, $\mathscr{B}(c) = (0.4, 0.3, 0.2)$. Define $\kappa = \{\tilde{0}, \tilde{1}, \mathscr{A}\}$ and $\kappa' = \{\tilde{0}, \tilde{1}, \mathscr{B}\}$ are NSTpS on X and Y. Let $\Lambda : (X, \kappa) \to (X', \kappa')$ be a map. Here $\Lambda(x) = a$, $\Lambda(y) = b$, $\Lambda(z) = c$. Assume $C(a) = (0.2, 0.4, 0.4)$, $C(b) = (0.2, 0.3, 0.5)$, $C(c) = (0.2, 0.3, 0.4)$ is a NSCS in X', then Λ^{-1} is a $N_{\alpha\psi}SCS$ in X. Hence Λ is a $N_{\alpha\psi}SCM$. To prove Λ^{-1} is a $N_{\alpha\psi}SCM$. Let $D(x) = (0.2, 0.3, 0.5)$, $D(y) = (0.2, 0.2, 0.6)$, $D(z) = (0.2, 0.4, 0.4)$ is a NSCS in X, then $\Lambda(D)$ is a $N_{\alpha\psi}SCS$ in X'. Thus Λ^{-1} is $N_{\alpha\psi}SCM$. Hence Λ is $N_{\alpha\psi}SHOM$.

Theorem 12 *Let $\Lambda : (X, \kappa) \to (X', \kappa')$ be a bijective $N_{\alpha\psi}SCM$, then the following statements are equivalent*

i. Λ *is* $N_{\alpha\psi}SCLM$
ii. Λ *is* $N_{\alpha\psi}SOM$
iii. Λ *is* $N_{\alpha\psi}SHOM$

Proof $(i) \to (ii)$. Let Λ is $N_{\alpha\psi}SCLM$ and Λ is bijective. Thus, Λ^{-1} is $N_{\alpha\psi}SCM$. It is clear that, for each NSOS in X, there is a $N_{\alpha\psi}SOS$ in X'. Hence Λ is $N_{\alpha\psi}SOM$.
$(ii) \to (iii)$. Given the map Λ is $N_{\alpha\psi}SCM$ and bijective. Then Λ^{-1} is $N_{\alpha\psi}SCM$. Thus, Λ and Λ^{-1} are $N_{\alpha\psi}SCM$. Hence Λ is $N_{\alpha\psi}SHOM$.
$(iii) \to (i)$. Let the map Λ is bijective. Clearly, Λ^{-1} exist. By hypothesis Λ is $N_{\alpha\psi}SHOM$. Then we have $N_{\alpha\psi}SCMs$ Λ and Λ^{-1}. Since, Λ^{-1} is a $N_{\alpha\psi}SCM$, then there exist $N_{\alpha\psi}SCS$ $\Lambda(\mathscr{A})$ in X' for each NSCS $\mathscr{A}$ in X.

Theorem 13 *If a bijective map Λ is a $N_{\alpha\psi}SIM$, then $\Lambda(N_{\alpha\psi}Scl(\mathscr{A})) = N_{\alpha\psi}Scl(\Lambda(\mathscr{A}))$.*

Proof Let $\mathscr{B}$ be a NS of X'. Assume Λ is a $N_{\alpha\psi}SIM$. Clearly, $\Lambda^{-1}(\mathscr{B})$ is a $N_{\alpha\psi}SCS$ in X for each $N_{\alpha\psi}SCS$ in X'. Since $\mathscr{B} \subseteq N_{\alpha\psi}Scl(\mathscr{B})$, $\Lambda^{-1}(\mathscr{B}) \subseteq \Lambda^{-1}(N_{\alpha\psi}Scl(\mathscr{B}))$, $N_{\alpha\psi}Scl(\Lambda^{-1}(\mathscr{B})) \subseteq N_{\alpha\psi}Scl(\Lambda^{-1}(N_{\alpha\psi}Scl(\mathscr{B}))) = \Lambda^{-1}(N_{\alpha\psi}Scl(\mathscr{B}))$. Since, $\Lambda^{-1}(\mathscr{B})$ is a $N_{\alpha\psi}SCS$ in X. Thus, $N_{\alpha\psi}Scl(\Lambda^{-1}(\mathscr{B})) \subseteq \Lambda^{-1}(N_{\alpha\psi}Scl(\mathscr{B})) \longrightarrow (1)$.

Let $\Lambda(\mathscr{B}) = \mathscr{A}$ is NS of X. Since Λ is bijective, inverse of Λ exist. Then Λ^{-1} is a $N_{\alpha\psi}SIM$. Clearly, $\Lambda^{-1}(\mathscr{B}) = \mathscr{A}$ is a $N_{\alpha\psi}SCS$ in X. This implies $N_{\alpha\psi}Scl(\mathscr{A})$ is a $N_{\alpha\psi}SCS$ in X. We know that, $\mathscr{A} \subseteq N_{\alpha\psi}Scl(\mathscr{A})$, $\Lambda(\mathscr{A}) \subseteq \Lambda(N_{\alpha\psi}Scl(\mathscr{A}))$, $N_{\alpha\psi}Scl(\Lambda(\mathscr{A})) \subseteq N_{\alpha\psi}Scl(\Lambda(N_{\alpha\psi}Scl(\mathscr{A}))) = \Lambda(N_{\alpha\psi}Scl(\mathscr{A}))$. $\Lambda^{-1}(N_{\alpha\psi}Scl(\mathscr{B})) \subseteq N_{\alpha\psi}Scl(\Lambda^{-1}(\mathscr{A})) \longrightarrow (2)$

From (1) and (2), we get $\Lambda(N_{\alpha\psi}Scl(\mathscr{A})) = N_{\alpha\psi}Scl(\Lambda(\mathscr{A}))$.

Theorem 14 *If a bijective map Λ is a $N_{\alpha\psi}SIM$, then $\Lambda(N_{\alpha\psi}Sint(\mathscr{B})) = N_{\alpha\psi}Sint(\Lambda(\mathscr{B}))$.*

Proof Proof is similar to Theorem 13.

6 Conclusion

$N_{\alpha\psi}SOM$, $N_{\alpha\psi}SCLM$, $N_{\alpha\psi}SCM$ and $N_{\alpha\psi}SIM$ are defined and their properties are studied. Finally, $N_{\alpha\psi}SHOM$ has also been established and investigated its properties.

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Invariant Study of Nets and Generalized Limits on Ideal Topological Spaces

Shyamapada Modak, Kulchhum Khatun, and Santanu Acharjee

Abstract The concept of limit point is an essential part of study in several areas of mathematics, viz., topology, functional analysis and metric space. Convergence of a net plays an important role in characterizing the continuity, Hausdorff space, metrizable space, etc. In this chapter, we are interested to study the preservation of limit points and its associated points. We develop two general methods of preservation of limit points.

1 Introduction

Many areas of mathematics study invariant operators in various manners. One of the invariant operators is idempotent operator. In topology, there are many generalized operators available which are invariant in nature. One of the operators is local function [15]. Local function [12, 13] can be defined depending on ideal. The notion of ideal was promoted in literature by Hamlett and Jankovic [15], but it was originated by Kuratowski [17]. Between 1933 and 1990, ideal was used only in a few papers. Some of them are [23, 28]. An ideal $\mathscr{I}$ [17] is a nonempty collection of subsets of X, which is closed with respect to finite union and hereditary property. If $\mathscr{I}$ is an ideal on X such that X is not a member of $\mathscr{I}$, then $\mathscr{I}$ is called a 'proper ideal' on X. If $\mathscr{I}$ is an ideal on X and τ is a topology on the same set X, then the triplicate $(X, \tau, \mathscr{I})$ is said to be an ideal topological space [8]. The condition $\tau \cap \mathscr{I} = \{\emptyset\}$ plays an important role for the study of an ideal topological space. An ideal topological space satisfying this condition is termed as Hayashi-Samuel space [8]. If $\mathscr{I}$ is an ideal on X of a topological space (X, τ), then the local function $(.)^* : \wp(X) \to \wp(X)$ is defined as $A^* = \{x \in X : U_x \cap A \notin \mathscr{I},\ U_x \in \tau(x)\}$, where $\tau(x)$ is the collection of

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all open sets of (X, τ) containing $x \in X$. According to the literature on topology, the notion A^* is same as $A^*(\mathscr{I})$ or $A^*(\mathscr{I}, \tau)$. A subset $A \subseteq X$ of an ideal topological space $(X, \tau, \mathscr{I})$ is called $*$-perfect [13] if $A = A^*$. For an ideal topological space $(X, \tau, \mathscr{I})$, the ψ operator [23] is defined as $\psi(A) = X \setminus (X \setminus A)^*$, for each $A \subseteq X$. A set-valued set function $f : \wp(X) \to \wp(X)$ is said to be associated [21] with a set-valued set function $g : \wp(X) \to \wp(X)$ if $f(A) = X \setminus g(X \setminus A)$, for each $A \in \wp(X)$. If a set-valued set function $f : \wp(X) \to \wp(X)$ is associated with a set-valued set function $g : \wp(X) \to \wp(X)$, then we say that f is associated with g and it is denoted as $f \sim^X g$. For an ideal topological space $(X, \tau, \mathscr{I})$, the local function $(.)^*$ is associated with the set operator ψ.

In this chapter, we are interested to evaluate the preservation set-valued set functions of various local functions and their complementary functions. This chapter also contains the convergence of the nets in terms of local function and its complementary function. Study of convergence of net or filter can be found in areas like metric space [26], normed function [26], topology [6, 17], generalized topology [7, 22] and supra topology [19]. Throughout this paper, we denote '*int*' for 'interior operator' and '*cl*' for 'closure operator' of a topological space.

2 Preliminaries

In this section, we discuss some preliminary definitions and notions which will be required in the next sections of this chapter.

Definition 1 A subset A of a topological space (X, τ) is said to be semi-open [18] (resp. preopen [11, 20], semi-preopen [4](=β-open [10]), b-open [5]) set, if $A \subseteq cl(int(A))$ (resp. $A \subseteq int(cl(A))$, $A \subseteq cl(int(cl(A)))$, $A \subseteq int(cl(A)) \cup cl(int(A)))$.

We denote the collection of all semi-open (resp. preopen, β-open, and b-open) sets containing $x \in X$ as $SO(X, x)$ (resp. $PO(X, x)$, $\beta O(X, x)$, $BO(X, x)$).

Definition 2 ([24]) Let $(X, \tau, \mathscr{I})$ be an ideal topological space. For a subset A of X, A^{*p} (resp. A^{*s}, $A^{*\beta}$, A^{*b})$= \{x \in X : U_x \cap A \notin \mathscr{I}$, for every $U_x \in PO(X, x)$ (resp. $SO(X, x)$, $\beta O(X, x)$, $BO(X, x))\}$.

Associated complementary set operators of the above set operators are defined as $\psi_p(A)$ (resp. $\psi_s(A)$, $\psi_\beta(A)$, $\psi_b(A)) = X\setminus(X\setminus A)^{*p}$ (resp. $(X\setminus A)^{*s}$, $(X\setminus A)^{*\beta}$, $(X\setminus A)^{*b}$). Let (X, τ) be a topological space. Then, for $A \subseteq X$, $scl(A)$ (resp. $pcl(A)$, $\beta cl(A)$, $bcl(A)) = \bigcap\{F \supseteq A : X\setminus F \in SO(X, \tau)$ $(resp.\ PO(X, \tau)$, $\beta O(X, \tau)$, $BO(X, \tau))\}$.

Definition 3 ([24]) Let $(X, \tau, \mathscr{I})$ be an ideal topological space. For a subset A of X, $\gamma_c(A)[1]$ $(resp.\ \gamma_{pc}(A)$, $\gamma_{sc}(A)$, $\gamma_{\beta c}(A)$, $\gamma_{bc}(A)) = \{x \in X : A \cap cl(U_x)$ (resp. $A \cap pcl(U_x)$, $A \cap scl(U_x)$, $A \cap \beta cl(U_x)$, $A \cap bcl(U_x)) \notin \mathscr{I}$, for every $U_x \in \tau(x)\}$.

Associated complementary set operators of the above set operators are defined as $\Gamma_c(A)$ (resp. $\Gamma_{pc}(A)$, $\Gamma_{sc}(A)$, $\Gamma_{\beta c}(A)$, $\Gamma_{bc}(A)) = X\backslash\gamma_c(X\backslash A)$ (resp. $\gamma_{pc}(X\backslash A)$, $\gamma_{sc}(X\backslash A)$, $\gamma_{\beta c}(X\backslash A)$, $\gamma_{bc}(X\backslash A)$).

Definition 4 ([24]) Let X be a set, $\mathscr{A} \subseteq \wp(X)$ and $\mathscr{I}$ be an ideal on X. A set operator $(.)^{c^*} : \wp(X) \to \wp(X)$ is defined as $(A)^{c^*}(\mathscr{A}, \mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I}, \text{ for every } U \in \mathscr{A}(x)\}$, where $\mathscr{A}(x) = \{U \in \mathscr{A} : x \in U\}$. Its associated complementary set operator is defined as $\psi_c(A) = X\backslash(X\backslash A)^{c^*}$.

Definition 5 ([27]) Let $(X, \tau, \mathscr{I})$ be an ideal topological space and γ be an operator associated with τ. For a subset A of X, $A^*_\gamma = \{x \in X : \gamma(U) \cap A \notin \mathscr{I} \text{ for every } U \in \tau(x)\}$.

Definition 6 ([2, 3]) Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A be a subset of X. Then, $A^{a^*} = \{x \in X : U \cap A \notin \mathscr{I}, \text{ for every } U \in \tau^a(x)\}$. Its associated complementary set operator is defined as $\mathfrak{R}_a(A) = X\backslash(X\backslash A)^{a^*}$.

Definition 7 ([25]) Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, (X, τ) is said to be an $\mathscr{I}_H$ space if for every pair of distinct points x and y in X, there exist two open sets U_1 and U_2 in τ such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 \in \mathscr{I}$.

3 Net Versus Local Function and ψ Operator

In this section, we study the convergence of a net to a point $x \in A^*$ (resp. ψ operator) of a set X. We know that $\{\emptyset\}$ and $\wp(X)$ are ideals on X. Then, $A^*(\{\emptyset\}) = cl(A)$ and $A^*(\wp(X)) = \emptyset$, for all $A \subseteq X$. If $\mathscr{I}_f$ is the ideal of finite subsets of X, then $A^*(\mathscr{I}_f) = A^\omega$ (set of all ω-accumulation points of $A \subseteq X$). If $\mathscr{I}_c$ is the ideal of countable subsets of X, then $A^*(\mathscr{I}_f)$ is the set of all condensation points of X. Now, we discuss the following theorems:

Theorem 1 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \in \wp(X)$. If $x \in A^*$, then there is a net S in A that converges to x.*

Proof Given that $x \in A^*$. Thus, for every $U \in \tau(x)$, $A \cap U \notin \mathscr{I}$. Thus, $A \cap U \neq \emptyset$. Define $\geq$ on $D = \{U \mid U \in \tau(x)\}$ by $U \geq V$ iff $U \subseteq V$. Then, $(D, \geq)$ is a directed set. Define a net $S : D \to A$ by $S(U) \in A \cap U$. Let G be an open set containing x. Then, G is also a member of the set D. Thus, for all $V \geq G$, $S(V) \in G$. Therefore, S converges to x. □

Converse of Theorem 1 is not true in general. We have the following example to justify it:

Example 1 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$, $\mathscr{I} = \{\emptyset, \{b\}\}$ and $A = \{b, c\}$. Then, $A^* = \{a, c\}$. We consider a directed set $D = (\wp(X), \subseteq)$ and a net $S : D \to A$ defined by $S(B) = b$, for all $B \in \wp(X)$. Here, S converges to b, but $b \notin A^*$.

Corollary 1 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \in \wp(X)$. If $x \notin \psi(A)$, then there is a net S in $X\backslash A$ converges to x.*

Proof If $x \notin \psi(A)$, then $x \in X\backslash\psi(A)$. It implies that $x \in (X\backslash A)^*$. Thus, from Theorem 1, there is a net S in $X\backslash A$ converges to x. □

Corollary 2 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \in \wp(X)$. If $x \notin \psi(X\backslash A)$, then there is a net S in A converges to x.*

Proof Proof is obvious from the fact that $\psi \sim^X *$ [21]. □

Theorem 2 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space, $A \subseteq X$ and a net $S : D \to A$ converges to $x \in X$. If $S(n) \cap A \notin \mathscr{I}$ for all $n \geq m$ where $m, n \in D$, then $x \in A^*$.*

Proof Given that $S : D \to A$ be a net converges to x. Then, for each $U_x \in \tau(x)$, there exists $m \in D$ such that $S(n) \in U_x$, for all $n \geq m$. Also, let $B_m = \{S(n) : n \in D \text{ and } n \geq m\}$. Then, $B_m \subseteq U_x$. Also, $B_m \subseteq A$. This implies that $B_m \subseteq U_x \cap A$. It is given that $S(n) \cap A \notin \mathscr{I}$, for all $n \geq m$. This implies that $B_m \cap A \notin \mathscr{I}$. Therefore, $B_m \notin \mathscr{I}$. Now, if possible suppose that $x \notin A^*$. Then, there exists $U_x \in \tau(x)$ such that $U_x \cap A \in \mathscr{I}$. This implies that $B_m \in \mathscr{I}$, a contradiction. Thus, $x \in A^*$. □

Corollary 3 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space, $A \subseteq X$ and a net $S : D \to A$ converges to $x \in X$. If $S(n) \cap A \notin \mathscr{I}$ for all $n \geq m$ where $m, n \in D$, then $x \notin \psi(X\backslash A)$.*

Proof Proof is obvious from the fact that $\psi \sim^X *$ [21]. □

Theorem 3 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If each net in X converges to unique limit, then (X, τ) is an $\mathscr{I}_H$ space.*

Proof Suppose $(X, \tau, \mathscr{I})$ be an ideal topological space and each net has a unique limit. If possible suppose that the space (X, τ) is not $\mathscr{I}_H$. Let $x, y \in X$ with $x \neq y$. Then, for all $U \in \tau(x)$ and for all $V \in \tau(y)$, $U \cap V \notin \mathscr{I}$. Define $\geq$ on $D = (\tau(x) \times \tau(y))$ by $(U_1, V_1) \geq (U_2, V_2)$ iff $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$. Then, $(D, \geq)$ is a directed set. Define a net $S : D \to X$ by $S(U, V) \in U \cap V$. Let $G \in \tau(x)$. Then, $(G, X) \in D$. If $(U, V) \geq (G, X)$, then $U \subseteq G$ and $S(U, V) \in U \cap V \subseteq U \subseteq G$. Thus, S converges to x. Similarly, S converges to y. But, this is a contradiction that each net has a unique limit. Thus, (X, τ) is an $\mathscr{I}_H$ space. □

For converse of Theorem 3, we have the following theorem:

Theorem 4 ([16]) *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If (X, τ) is an $\mathscr{I}_H$ space, then, for each $x \in X$, there is a $V \in \tau(y)$ such that $x \notin V^*$, where $x \neq y$.*

Corollary 4 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If (X, τ) is an $\mathscr{I}_H$ space, then, for each $x \in X$, there is a $V \in \tau(y)$ such that $x \in \psi(X\backslash V)$, where $x \neq y$.*

Proof Proof is obvious from the fact that $\psi \sim^X *$ [21]. □

Corollary 5 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If (X, τ) is an $\mathscr{I}_H$ Hayashi-Samuel space, then, for each $x \in X$, there is a $V \in \tau(y)$ such that $x \notin cl(V)$, where $x \neq y$.*

Proof It is given that (X, τ) is an $\mathscr{I}_H$ space. Then, for any $x,\ y \in X$ with $x \neq y$, there exist $U \in \tau(x)$ and $V \in \tau(y)$ such that $U \cap V \in \mathscr{I}$. Due to the definition of Hayashi-Samuel space, if $U \cap V \neq \emptyset$, then $U \cap V \notin \tau$ but $U \cap V \in \tau$. Thus, $U \cap V = \emptyset$. Hence, $x \notin cl(V)$. □

Corollary 6 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If (X, τ) is an $\mathscr{I}_H$ Hayashi-Samuel space, then, for each $x \in X$, there is a $V \in \tau(y)$ such that $x \in int(X \backslash V)$, where $x \neq y$.*

Proof Proof is obvious from the fact that $cl \sim^X int$ [21]. □

4 Preservation of Generalized Limit Points

In this section, we introduce g-preserving set-valued set function and related properties.

Definition 8 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, a set-valued set function $f : \wp(X) \to \wp(X)$ is called g-preserving if for any $A \subset X$, $(f(A))^g = A^g$.

From the literature on topology, known g may be the following operators: $(.)^*$ [28], $(.)^{*_b}$ [24], $(.)^{*_\beta}$ [24], $(.)^{*_s}$ [24], $(.)^{*_p}$ [24], $(.)^{a^*}$ [2, 3], γ_c [1], γ_{sc} [24], $\gamma_{\beta c}$ [24], γ_{bc} [24], $(.)^{c^*}$ and γ_{pc} [24] and $(.)^*_\gamma$ [27].

The operators $\gamma_c, \gamma_{sc}, \gamma_{\beta c}, \gamma_{bc}, (.)^{c^*}, \gamma_{pc}$ and $(.)^*_\gamma$ are also defined in terms of closure neighbourhoods of a point. Furthermore, the above operators have some common properties [24]. However, we only focus on the local function. Local function is a compact form of the various types of limit points. Suppose Ψ_g is the associated set-valued set function, and it is of the form $\Psi_g(A) = X \backslash (X \backslash A)^g$. Hence, $\Psi_g \sim^X (.)^g$. We procure the following example:

Example 2 Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $I \in \mathscr{I}$. Then,

(1) $f : \wp(X) \to \wp(X)$ defined as $f(A) = A \cup I$ is a g-preserving set-valued set function,
(2) $f : \wp(X) \to \wp(X)$ defined as $f(A) = A \backslash I$ is a g-preserving set-valued set function,
(3) for g-preserving set-valued set function $f : \wp(X) \to \wp(X)$, $f_1 : \wp(X) \to \wp(X)$ defined as $f_1(A) = f(A) \cup A$ is also a g-preserving set-valued set function,
(4) $cl^* : \wp(X) \to \wp(X)$ defined as $cl^*(A) = A \cup A^g$ is a g-preserving set-valued set function.

Proof:

(1) From [28], we know that $(f(A))^g = (A \cup I)^g = A^g$. Hence, f is a g-preserving set-valued set function.
(2) From [28], we know that $(f(A))^g = (A \backslash I)^g = A^g$. Hence, f is a g-preserving set-valued set function.
(3) Since f is a g-preserving set-valued set function, thus $(f(A))^g = A^g$. From [28], $(f_1(A))^g = (f(A) \cup A)^g = (f(A))^g \cup A^g = A^g \cup A^g = A^g$. Thus, f_1 is a g-preserving set-valued set function.
(4) $(cl^*(A))^g = (A \cup A^g)^g = A^g \cup (A^g)^g = A^g$ as $(A^g)^g \subseteq A^g$. Thus, cl^* is a g-preserving set-valued set function.

Theorem 5 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then,*

(1) composition of two g-preserving set-valued set functions is a g-preserving set-valued set function,
(2) for identity set-valued set function $\mathrm{T}(A) = A$, $\mathrm{T} \circ f = f \circ \mathrm{T} = f$ holds, where f is a g-preserving set-valued set function on $\wp(X)$,
(3) for g-preserving set-valued set function $f : \wp(X) \to \wp(X)$, f^n is also g-preserving, for all $n \in \mathbb{N}$.

Remark 1 It is interesting to find that there is not any g-preserving set-valued set function f_1 on $\wp(X)$ such that $f \circ f_1 = f_1 \circ f = \mathrm{T}$. If f is a g-preserving set-valued set function on $\wp(X)$ and any subset A of X is g dense-in-itself [9], then $(f(A))^g = \mathrm{T}(A)$.

Theorem 6 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, $f : \wp(X) \to \wp(X)$ is g-preserving if and only if $(f(A))^g = X \backslash \Psi_g(X \backslash A)$.*

Proof Let $f : \wp(X) \to \wp(X)$ be g-preserving. We know that $\Psi_g(A) = X \backslash (X \backslash A)^g$. Then, $\Psi_g(X \backslash A) = X \backslash A^g$. It implies $X \backslash \Psi_g(X \backslash A) = A^g = (f(A))^g$. Hence, $(f(A))^g = X \backslash \Psi_g(X \backslash A)$.

Conversely, $\Psi_g(A) = X \backslash (X \backslash A)^g$. Then, $\Psi_g(X \backslash A) = X \backslash A^g$. It yields $X \backslash \Psi_g(X \backslash A) = A^g$. Thus, $(f(A))^g = A^g$. Hence, $f : \wp(X) \to \wp(X)$ is g-preserving. □

Remark 2 Suppose union of two set-valued set functions $f_1, f_2 : \wp(X) \to \wp(X)$ is defined as $(f_1 \cup f_2)(A) = f_1(A) \cup f_2(A)$. Thus, for g-preserving f_1 and f_2, $f_1 \cup f_2$ is also g-preserving.

Proof From [28], we can find that $((f_1 \cup f_2)(A))^g = (f_1(A) \cup f_2(A))^g = (f_1(A))^g \cup (f_2(A))^g = A^g \cup A^g = A^g$. Thus, for g-preserving f_1 and f_2, $f_1 \cup f_2$ is also g-preserving. □

Theorem 7 *Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be two ideals on the set X of a topological space (X, τ). If f is g-preserving with respect to the both ideals $\mathscr{I}_1$ and $\mathscr{I}_2$, then f is g-preserving with respect to the ideal $\mathscr{I}_1 \cap \mathscr{I}_2$.*

Proof From [28], we get $(f(A))^g(\mathscr{I}_1 \cap \mathscr{I}_2, \tau) = (f(A))^g(\mathscr{I}_1, \tau) \cup (f(A))^g(\mathscr{I}_2, \tau) = A^g(\mathscr{I}_1, \tau) \cup A^g(\mathscr{I}_2, \tau) = A^g(\mathscr{I}_1 \cap \mathscr{I}_2, \tau)$. Hence, f is g-preserving with respect to the ideal $\mathscr{I}_1 \cap \mathscr{I}_2$. □

Converse of the above theorem is not true in general. We give the following example to justify our claim:

Example 3 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$, $\mathscr{I}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathscr{I}_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then, $\mathscr{I}_1 \cap \mathscr{I}_2 = \{\emptyset, \{a\}\}$. Hence, we have the table given below for different sets considering as A. This example shows that f is g-preserving with respect to the ideals $\mathscr{I}_1 \cap \mathscr{I}_2$ and $\mathscr{I}_2$ but it is not g-preserving with respect to the ideal $\mathscr{I}_1$.

A	$A^g(\tau, \mathscr{I}_1)$	$A^g(\tau, \mathscr{I}_2)$	$A^g(\tau, \mathscr{I}_1 \cap \mathscr{I}_2)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{b\}$	$\emptyset$	X	X
$\{c\}$	$\{c\}$	$\emptyset$	$\{c\}$
$\{a, b\}$	$\emptyset$	X	X
$\{a, c\}$	$\{c\}$	$\emptyset$	$\{c\}$
$\{b, c\}$	$\{c\}$	X	X
X	$\{c\}$	X	X

Let us define a set-valued set function $f : \wp(X) \to \wp(X)$ by

$$f(A) = \begin{cases} \emptyset, & \text{if } A = \emptyset; \{a\}, \\ X, & \text{if } A = X, \\ \{c\}, & \text{if } A = \{a, c\}, \\ \{a, c\}, & \text{if } A = \{c\}, \\ \{b\}, & \text{if } A = \{a, b\}; \{b, c\}, \\ \{b, c\}, & \text{if } A = \{b\}. \end{cases}$$

Theorem 8 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $f : \wp(X) \to \wp(X)$ be a g-preserving set-valued set function, then $f(A) \notin \mathscr{I}$, for all nonempty $*$-perfect subset A of X.*

Proof Let $f(A) \in \mathscr{I}$, where A is a nonempty $*$-perfect subset of X. Then, $(f(A))^g$ is nonempty as f is a g-preserving set-valued set function. Let $x \in (f(A))^g$. Then, $U \cap f(A) \notin \mathscr{I}$, for each $U \in \tau(x)$. If $f(A) \in \mathscr{I}$, then $U \cap f(A) \in \mathscr{I}$ as $U \cap f(A) \subseteq f(A)$, a contradiction. Hence, $f(A) \notin \mathscr{I}$. □

This theorem restricts us for defining the g-preserving set-valued set function. Converse statement of the above theorem is not true in general. We have the following example:

Example 4 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$, $\mathscr{I} = \{\emptyset, \{a\}\}$. We define a set-valued set function $f : \wp(X) \to \wp(X)$ as given below:

$$f(A) = \begin{cases} X, & \text{if } A = X; \emptyset; \{a\}, \\ \{b\}, & \text{if } A = \{a, c\}, \\ \{a, c\}, & \text{if } A = \{b\}, \\ \{c\}, & \text{if } A = \{a, b\}; \{b, c\}, \\ \{b, c\}, & \text{if } A = \{c\}. \end{cases}$$

In this example, $f(A) \notin \mathscr{I}$, but f is not a g-preserving set-valued set function.

5 L^g-Preservation

Selim and Modak [21] introduced the notion of associated function. Using associated function, Modak and Hoque [14] developed a new concept for locally closed sets and defined L^g operator on an ideal topological space.

Definition 9 Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, we define the operator $L^g : \wp(X) \to \wp(X)$ as $L^g(A) = A^g \cap \Psi_g(A)$, for $A \in \wp(X)$. Now, a set-valued set function $f : \wp(X) \to \wp(X)$ is called L^g-preserving if $f(L^g(A)) = L^g(A)$.

Our known g and Ψ_g may be the following operators: $(.)^*$ and ψ [23], $(.)^{*_b}$ and ψ_b [24], $(.)^{*_\beta}$ and ψ_β [24], $(.)^{*_s}$ and ψ_s [24], $(.)^{*_p}$ and ψ_p [24], $(.)^{a^*}$ and $\mathfrak{R}_a$ [2, 3], γ_c and Γ_c [24], γ_{sc} and Γ_{sc} [24], $\gamma_{\beta c}$ and $\Gamma_{\beta c}$ [24], γ_{bc} and Γ_{bc} [24], $(.)^{c^*}$ and ψ_c [24], γ_{pc} and Γ_{pc} [24], respectively. In general, the operators local function and its associated complementary set operator obey some common properties [24] for any collection.

In a topological space (X, τ), interior and closure operators are associated to each other. So, we can write as $int \sim^X cl$. For a subset A of X, $int(A) \cap cl(A) = int(A)$. In this case, L^g-preserving set-valued set function plays the role only on interior operator. Thus, the study of L^g-preserving set-valued set function may be more interesting if we consider ideal topological spaces. It is also noticed that $(.)^g$ is not a closure operator and Ψ_g is not an interior operator. But, these two operators induce closure and interior operators through the following ways: for a subset A of X in the ideal topological space $(X, \tau, \mathscr{I})$, $\mathrm{h}(A) = A \cup A^g$ is a closure operator and $\mathrm{j}(A) = A \cap \Psi_g(A)$ is an interior operator. Both the operators induce the same topology and it is called the $*$-topology [12] (or g-topology) on X. The sets $A^g(\tau)$ and $A^g(\tau^g)$ are same. Again, the sets $\Psi_g(A)(\tau)$ and $\Psi_g(A)(\tau^g)$ are same.

Theorem 9 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then,*

(1) composition of two L^g-preserving set-valued set functions is a L^g-preserving set-valued set function,
(2) for identity set-valued set function $\mathrm{T}(A) = A$, $\mathrm{T} \circ f = f \circ \mathrm{T} = f$ holds, where f is a L^g-preserving set-valued set function on $\wp(X)$.

In the following two examples, we show that g-preserving does not imply L^g-preserving and vice versa.

Example 5 Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{b\}, \{a, b\}, \{b, c\}\}$ and $\mathscr{I} = \{\varnothing, \{a\}\}$. Then, we have the following table:

A	A^g	$\Psi_g(A)$	$L^g(A)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{b\}$	X	$\{a, b\}$	$\{a, b\}$
$\{c\}$	$\{c\}$	$\emptyset$	$\emptyset$
$\{a, b\}$	X	$\{a, b\}$	$\{a, b\}$
$\{a, c\}$	$\{c\}$	$\emptyset$	$\emptyset$
$\{b, c\}$	X	X	X
X	X	X	X

According to Example 3, f is g-preserving with respect to the ideal $\mathscr{I}$, but here we see that it is not L^g-preserving with respect to the ideal $\mathscr{I}$. Since, $f(L^g(A)) = f(\{a, b\}) = \{b\}$, for $A = \{b\}, \{a, b\}$. So, $f(L^g(A)) \neq L^g(A)$.

Example 6 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$ and $\mathscr{I} = \{\emptyset, \{a\}\}$. We define a set-valued set function $f : \wp(X) \to \wp(X)$ as follows:

$$f(A) = \begin{cases} \emptyset, & \text{if } A = \emptyset; \{a\}, \\ X, & \text{if } A = X, \\ \{b\}, & \text{if } A = \{a, c\}, \\ \{a, c\}, & \text{if } A = \{c\}, \\ \{a, b\}, & \text{if } A = \{a, b\}, \\ \{c\}, & \text{if } A = \{b, c\}, \\ \{b, c\}, & \text{if } A = \{b\}. \end{cases}$$

In this example, $f(L^g(A)) = L^g(A)$, for all $A \in \wp(X)$. Thus, f is L^g-preserving but it is not a g-preserving set-valued set function with respect to the ideal $\mathscr{I}$.

Theorem 10 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, the following statements are equivalent:*

(1) $f : \wp(X) \to \wp(X)$ is a L^g-preserving set-valued set function,
(2) for each $A \subseteq X$, $f(L^g(A)) = \Psi_g(A) \backslash \Psi_g(X \backslash A)$,
(3) for each $A \subseteq X$, $f(L^g(A)) = A^g \backslash (X \backslash A)^g$.

Proof (1) $\Rightarrow$ (2): $f(L^g(A)) = L^g(A) = A^g \cap \Psi_g(A) = (X\backslash\Psi_g(X\backslash A)) \cap \Psi_g(A) = \Psi_g(A)\backslash\Psi_g(X\backslash A)$.
(2) $\Rightarrow$ (3): $f(L^g(A)) = \Psi_g(A)\backslash\Psi_g(X\backslash A) = (X\backslash(X\backslash A)^g)\backslash(X\backslash A^g) = A^g\backslash(X\backslash A^g)$.
(2) $\Rightarrow$ (1): $f(L^g(A)) = A^g\backslash(X\backslash A)^g = A^g\backslash(X\backslash\Psi_g(A)) = A^g \cap (X\backslash(X\backslash\Psi_g(A))) = A^g \cap \Psi_g(A) = L^g(A)$. □

Theorem 11 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $f : \wp(X) \to \wp(X)$ be a homeomorphism and $g : \wp(X) \to \wp(X)$ be a L^g-preserving set-valued set function, then $f[g(L^g(A))] = L^g(f(A))$, for all $A \in \wp(X)$.*

Proof Since g is L^g-preserving, so $f[g(L^g(A))] = f(L^g(A)) = f(A^g \cap \Psi_g(A)) = f(A^g) \cap f(\Psi_g(A)) = (f(A))^g \cap \Psi_g(f(A)) = L^g(f(A))$. □

Theorem 12 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then, the following statements are equivalent:*

(1) the space $(X, \tau, \mathscr{I})$ is a Hayashi-Samuel space,
(2) if $f : \wp(X) \to \wp(X)$ is a L^g-preserving set-valued set function, then $f(L^g(A)) = \Psi_g(A)$, for all $A \subseteq X$.

Proof (1) $\Rightarrow$ (2): Since $(X, \tau, \mathscr{I})$ is a Hayashi-Samuel space, so $f(L^g(A)) = L^g(A) = \Psi_g(A) \cap A^g = \Psi_g(A)$.

(2) $\Rightarrow$ (1): Since f is L^g-preserving and $f(L^g(A)) = \Psi_g(A)$, thus $L^g(A) = \Psi_g(A)$. Therefore, $\Psi_g(A) \cap A^g = \Psi_g(A)$. Hence, $\Psi_g(A) \subseteq A^g$, for all $A \subseteq X$. So, take $A = X$, then $\Psi_g(X) \subseteq X^g$. Therefore, $X \subseteq X^g$. Hence, X is a Hayashi-Samuel space. □

Theorem 13 *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If $f : \wp(X) \to \wp(X)$ defined as $f(A) = X\backslash A$ is a L^g-preserving set-valued set function, then the following results hold:*

(1) $A^g \cap \Psi_g(A) = (X\backslash A^g) \cup (X\backslash\Psi_g(A))$,
(2) $(X\backslash A)^g \cap \Psi_g(X\backslash A) = A^g \cup \Psi_g(A)$.

Proof (1) Since f is L^g-preserving, so $L^g(A) = f(L^g(A)) = X\backslash L^g(A) = X\backslash(A^g \cap \Psi_g(A)) = (X\backslash A^g) \cup (X\backslash\Psi_g(A))$. Thus, $A^g \cap \Psi_g(A) = (X\backslash A^g) \cup (X\backslash\Psi_g(A))$.

(2) Since f is L^g-preserving, so $L^g(X\backslash A) = f(L^g(X\backslash A)) = X\backslash L^g(X\backslash A) = X\backslash((X\backslash A)^g \cap \Psi_g(X\backslash A)) = (X\backslash(X\backslash A)^g) \cup (X\backslash\Psi_g(X\backslash A)) = A^g \cup \Psi_g(A)$. Thus, $(X\backslash A)^g \cap \Psi_g(X\backslash A) = A^g \cup \Psi_g(A)$. □

Now, it is important to note that (i) $A^g = cl(A)$ and $\Psi_g(A) = int(A)$, for $\mathscr{I} = \{\emptyset\}$; and (ii) $A^g = \emptyset$ and $\Psi_g(A) = X$, for $\mathscr{I} = \wp(X)$.

6 Conclusion

In this paper, properties of associated functions are studied by introducing g-preserving set-valued set function and L^g-preserving set-valued set function. Moreover, several results are provided for ideal topological space which are simultaneously Hayashi-Samuel space. At last, this paper has potential for the extension in topological graph theory.

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On Associated Functions and Filters

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Abstract Recently Modak and Selim introduced associated functions and studied some basic results. In this chapter, we investigate some more results of associated functions. We also establish relationship of associated functions with semigroup structures. Moreover, we show that associated functions can produce two hypergraph structures and we prove some related results.

1 Introduction

Topology is a branch of mathematics that has connections to almost all the areas of science and humanities. Due to the interconnecting nature of topology, many generalized ideas have been developed so far in topology. Ideal [7, 10], grill [6], etc. are some of the generalised structures in topology that are highly used. One of the recent generalised ideas of topology is *associated function*. In 2021, Modak and Selim [14] introduced associated functions and studied some of their properties. Associated function is a generalized idea of the standard results: $int(A) = X \setminus cl(X \setminus A)$ [10], $\psi(A) = X \setminus (X \setminus A)^*$ [15], $\mathscr{F} - int(A) = X \setminus \mathscr{F} - cl(X \setminus A)$ [13] and many others. These relations have been defined in terms of mathematical structures using ideal, filter, etc. But, the associated functions [14] are beyond existing topological operators, viz., operators stated in [10, 13, 15] and they are closely related to set theoretical operators.

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Although one may find that the closure and interior operators are not new in general topology, yet several recent research articles are available on these two topics. Recently, Jung and Nam [8] investigated some generalized properties of interior and closure operators in general topology. Kumar et al. [9] investigated compactness and subspaces in multiset topology using closure and interior operators. Shravan and Tripathy [18] used closure and interior operators to study multiset mixed topology. In 2022, Asici [1] constructed t-norms and t-conorms in bounded lattices by using interior and closure. Similarly, Zhao and Pang [22] investigated L-closure and L-interior operators in L-fuzzy set. A rigorous study on bases of closure operators in complete lattices was done in [19]. Another study on bounded lattices was done in [5]. Lei and Zhang [11] investigated closure systems and semantics. In 2019, Baltag et al. [3] showed that belief system is connected to closure as well as interior operators of topology. Li and Zhang [12] investigated several properties of granulation and evidence theory using closure and interior operators in a finite topological space. It is also important to find that Polat [16] studied key agreement protocol using closure and interior operators of topology. Since, we know that closure and interior operators of general topology can be connected by the relation $int(A) = X \setminus cl(X \setminus A)$ [21], thus it is easy to understand that associated functions may open several new directions of research in near future because these aforementioned relations between closure and interiors along with other two are generalisations of associated functions. Moreover, significance of this work can be found in set theory, semigroup theory, topology, hypergraphs, and social networking.

This chapter has four main sections. In Sect. 2, we discuss several new ideas of associated functions. In Sect. 3, a few new operators are introduced with associated functions. Filter convergence due to associated functions are discussed in Sect. 4. In Sect. 5, we discuss hypergraph structures using associated functions.

2 Associated Functions

In this section, we discuss some properties of associated functions. We also show that associated functions can be connected to semigroup theory.

Definition 1 ([14]) A set-valued set function $f : \mathscr{P}(X) \to \mathscr{P}(X)$ is said to be *associated* with a set-valued set function $g : \mathscr{P}(X) \to \mathscr{P}(X)$ if $f(A) = X \setminus g(X \setminus A)$ for each $A \in \mathscr{P}(X)$. It is denoted as $f \sim^X g$.

Theorem 1 *$f \sim^X g$ if and only if $g \sim^X f$.*

Proof Necessary part is obvious from [14]. For sufficient part, we consider $g \sim^X f$. Given that $g(A) = X \setminus f(X \setminus A)$, for all $A \in \mathscr{P}(X)$. This implies that $f(X \setminus A) = X \setminus g(A)$. Thus, we have $f(A) = X \setminus g(X \setminus A)$, for all $A \in \mathscr{P}(X)$. Therefore, $f \sim^X g$. □

Remark 1 If we consider $\sim^X$ as a relation on a set of associated functions, then $\sim^X$ is reflexive, symmetric but not transitive. So, $\sim^X$ is not an equivalence relation.

For transitivity, we may consider the following result of [14]:

Theorem 2 ([14]) *If $f \sim^X g$ and $g \sim^X h$, then $f \equiv h$.*

Suppose $f \sim^X g$. If there exists a set-valued set function $l : \mathscr{P}(X) \to \mathscr{P}(X)$ such that $l \sim^X g$ (resp. $f \sim^X l$), then we must have $f \equiv l$ (resp. $l \equiv g$). Thus, we conclude that there are no two distinct set-valued set functions associated with one set-valued set function simultaneously. For converse part of Theorem 2, we have the following theorem:

Theorem 3 *Let $f, h : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions. If $f \equiv h$, then there exists a set-valued set function $g : \mathscr{P}(X) \to \mathscr{P}(X)$ such that $f \sim^X g$ and $g \sim^X h$.*

Proof Given that $f(A) = h(A)$, for all $A \in \mathscr{P}(X)$. Suppose $g : \mathscr{P}(X) \to \mathscr{P}(X)$ be a set-valued set function defined by $f(A) = X \setminus g(X \setminus A)$. Then, $f \sim^X g$. As $f \equiv h$, then $h \sim^X g$ holds. Hence, from Theorem 1, $g \sim^X h$. □

Remark 2 $\sim^X$ is not an anti-symmetric relation on X.

Theorem 4 *Let $f, g, h, k : \mathscr{P}(X) \to \mathscr{P}(X)$ be set-valued set functions such that $f \sim^X g$ and $h \sim^X k$. Then,*

(1) $f \circ h \sim^X g \circ k$,
(2) $f \circ k \sim^X g \circ h$.

Proof Let $A \in \mathscr{P}(X)$. Then,

(1) $(f \circ h)(A) = f[X \setminus k(X \setminus A)] = X \setminus (g \circ k)(X \setminus A)$. Thus, $f \circ h \sim^X g \circ k$.

(2) $(f \circ k)(A) = f[X \setminus h(X \setminus A)] = X \setminus (g \circ h)(X \setminus A)$. Thus, $f \circ k \sim^X g \circ h$. □

Remark 3 Let $Asso(F^X)$ be the collection of all associated functions defined on power set of X. Then, $(Asso(F^X), \circ)$ is a semigroup. Further, the set-valued set function $I : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $I(A) = A$, for all $A \in \mathscr{P}(X)$ is associated with I, i.e., $I \sim^X I$, and I is the identity of $(Asso(F^X), \circ)$.

Theorem 5 *Let X be a set, $f \sim^X g$ and $A \in \mathscr{P}(X)$. Then, $A \subseteq f(A)$ if and only if $g(X \setminus A) \subseteq X \setminus A$.*

Proof Given that $f(A) = X \setminus g(X \setminus A)$. Let $A \subseteq f(A)$, then $A \subseteq X \setminus g(X \setminus A)$. Thus, $g(X \setminus A) \subseteq X \setminus A$. Hence, we prove the theorem.

Similarly, we can prove the converse part. □

Corollary 1 *Let $f \sim^X g$ and $A \in \mathscr{P}(X)$. Then, $A \subseteq g(A)$ if and only if $f(X \setminus A) \subseteq X \setminus A$.*

Proof The proof follows from Theorems 1 and 5. □

Theorem 6 *Let $f : \mathscr{P}(X) \to \mathscr{P}(X)$ be a set-valued set function and $A \subseteq f(A)$, for each $A \in \mathscr{P}(X)$. Then, there exists a set-valued set function $g : \mathscr{P}(X) \to \mathscr{P}(X)$ such that $f \sim^X g$ and $g(A) \subseteq A$, for each $A \in \mathscr{P}(X)$.*

Proof Put $g(A) = X \setminus f(X \setminus A)$. Then, $f \sim^X g$. Given that $X \setminus A \subseteq f(X \setminus A)$. Then, $A \supseteq [X \setminus f(X \setminus A)]$ implies $g(A) \subseteq A$. □

Theorem 7 *Let $f \sim^X g$. Then, f is idempotent if and only if g is idempotent.*

Proof Suppose $f \sim^X g$ holds and assume that $(f \circ f)(A) = A$. Now, $(f \circ f)(A) = f[X \setminus g(X \setminus A)] = X \setminus g[g(X \setminus A)]$. This implies that $A = X \setminus g[g(X \setminus A)]$. So, $X \setminus A = (g \circ g)(X \setminus A)$ and this true for all $A \in \mathscr{P}(X)$. Hence, $A = (g \circ g)(A)$.

Converse part follows from Theorem 1. Hence, the theorem is proven. □

In view of the above theorem, we give the following example:

Example 1 Let X be any non-empty set. Define $f : \mathscr{P}(X) \to \mathscr{P}(X)$ by $f(A) = X \setminus A$. Then, $(f \circ f)(A) = A$. Thus, f is idempotent if and only if g is idempotent.

Corollary 2 *Let $f : \mathscr{P}(X) \to \mathscr{P}(X)$ be a set-valued set function. If f is idempotent, then there exists an idempotent set-valued set function $g : \mathscr{P}(X) \to \mathscr{P}(X)$ such that $f \sim^X g$.*

Proof Let $g(A) = X \setminus f(X \setminus A)$. Then, $f \sim^X g$. Other part follows from Theorem 7. □

Example 2 (1) Let X be any non-empty set and B be a fixed subset of X. Define $f : \mathscr{P}(X) \to \mathscr{P}(X)$ by $f(A) = A \cup B$. Then, $f(f(A)) = f(A) \cup B = (A \cup B) \cup B = A \cup B = f(A)$. In this case, we will take $g(A) = X \setminus f(X \setminus A) = X \setminus [(X \setminus A) \cup B] = [X \setminus (X \setminus A)] \cap (X \setminus B) = A \cap (X \setminus B) = A \setminus B$. Hence, $g(g(A)) = (A \setminus B) \setminus B = g(A)$.

(2) Let X be any non-empty set and D be a fixed subset of X. Define $h : \mathscr{P}(X) \to \mathscr{P}(X)$ by $h(A) = A \cap D$. Then, $h(h(A)) = h(A) \cap D = (A \cap D) \cap D = A \cap D = h(A)$. Define $k : \mathscr{P}(X) \to \mathscr{P}(X)$ by $k(A) = X \setminus h(X \setminus A)$. Then, $k(A) = X \setminus [(X \setminus A) \cap D] = [X \setminus (X \setminus A)] \cup (X \setminus D) = A \cup (X \setminus D)$. Now, $k(k(A)) = k(A) \cup (X \setminus D) = [A \cup (X \setminus D)] \cup (X \setminus D) = A \cup (X \setminus D)$.

Definition 2 Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions. Define $f \cup g : \mathscr{P}(X) \to \mathscr{P}(X)$ by $(f \cup g)(A) = f(A) \cup g(A)$.

It is obvious that $f \cup g = g \cup f$.

Definition 3 Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions. Define $f \cap g : \mathscr{P}(X) \to \mathscr{P}(X)$ by $(f \cap g)(A) = f(A) \cap g(A)$.

It is obvious that $f \cap g = g \cap f$.

Corollary 3 *Let $f \sim^X g$. Then, for any $A \in \mathscr{P}(X)$, $(f \cap g)(A) = g(A) \setminus g(X \setminus A) = f(A) \setminus f(X \setminus A)$.*

Proof $(f \cap g)(A) = f(A) \cap g(A) = [X \setminus g(X \setminus A)] \cap g(A) = g(A) \setminus g(X \setminus A)$.

For second relation, we have $(f \cap g)(A) = f(A) \cap [X \setminus f(X \setminus A)] = f(A) \setminus f(X \setminus A)$. □

Theorem 8 *Let $f \sim^X g$ and $h \sim^X k$. Then,*
(1) $f \cup g \sim^X f \cap g$,
(2) $f \cup h \sim^X g \cap k$,
(3) $f \cap h \sim^X g \cup k$.

Proof For $A \in \mathscr{P}(X)$, $(f \cup g)(A) = f(A) \cup g(A) = [X \setminus g(X \setminus A)] \cup [X \setminus f(X \setminus A)] = X \setminus [f(X \setminus A) \cap g(X \setminus A)] = X \setminus (f \cap g)(X \setminus A)$. Hence, $f \cup g \sim^X f \cap g$.

We can prove other results by similar ways. Now, from Theorem 8, it is obvious that $(Asso(F^X), \cup)$ and $(Asso(F^X), \cap)$ are semigroups. □

Example 3 Let X be non-empty set. Define $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ by the following:

$$f(A) = X, \text{ for all } A \in \mathscr{P}(X),$$
$$g(A) = \emptyset, \text{ for all } A \in \mathscr{P}(X).$$

Then, $f \sim^X g$.

Remark 4 For the semigroup $(Asso(F^X), \cup)$, g defined in Example 3 is the identity element. Further, for the semigroup $(Asso(F^X), \cap)$, f defined in Example 3 is the identity element.

Definition 4 Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions. Define $f \setminus g : \mathscr{P}(X) \to \mathscr{P}(X)$ by $(f \setminus g)(A) = f(A) \setminus g(A)$, for all $A \in \mathscr{P}(X)$.

Theorem 9 *Let $f \sim^X g$. Then, for any $A \in \mathscr{P}(X)$, $(f \setminus g)(A) = (f \setminus g)(X \setminus A)$.*

Proof $(f \setminus g)(A) = f(A) \setminus g(A) = [X \setminus g(X \setminus A)] \setminus [X \setminus f(X \setminus A)] = f(X \setminus A) \setminus g(X \setminus A) = (f \setminus g)(X \setminus A)$. □

Remark 5 In Example 3, $f \setminus g \neq g \setminus f$.

Definition 5 Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions. Define $f \Delta g : \mathscr{P}(X) \to \mathscr{P}(X)$ by $(f \Delta g)(A) = [f(A) \setminus g(A)] \cup [g(A) \setminus f(A)]$, for all $A \in \mathscr{P}(X)$.

Theorem 10 *Let $f \sim^X g$. Then, for $A \in \mathscr{P}(X)$, $(f \Delta g)(A) = (f \Delta g)(X \setminus A)$ and $(g \Delta f)(A) = (g \Delta f)(X \setminus A)$.*

Proof $(f \Delta g)(A) = [f(A) \setminus g(A)] \cup [g(A) \setminus f(A)] = (f \setminus g)(A) \cup (g \setminus f)(A) = [(f \setminus g)(X \setminus A)] \cup [(g \setminus f)(X \setminus A)] = (f \Delta g)(X \setminus A)$. □

Definition 6 Let $f : \mathscr{P}(X) \to \mathscr{P}(X)$ and $A \in \mathscr{P}(X)$. Define $f|_A : \mathscr{P}(X) \to \mathscr{P}(X)$ by $f|_A(B) = f(B) \cap A$, for all $B \in \mathscr{P}(X)$.

Theorem 11 *Let $f \sim^X g$. Then, $f|_A \sim^A g|_A$.*

Proof $(f|_A)(B) = f(B) \cap A = [X \setminus g(X \setminus B)] \cap A = A \setminus g(X \setminus B)$. Therefore, $A \setminus f|_A(B) = g(X \setminus B)$. Thus, $[A \setminus f|_A(B)] \cap A = g(X \setminus B) \cap A$. So, $A \setminus f|_A(B) = g|_A(X \setminus B)$ and hence, $f|_A \sim^A g|_A$. □

Definition 7 Let $f : \mathscr{P}(X) \to \mathscr{P}(X)$ and $g : \mathscr{P}(Y) \to \mathscr{P}(Y)$ be two set-valued set functions. Define $(f \times g) : \mathscr{P}(X) \times \mathscr{P}(Y) \to \mathscr{P}(X) \times \mathscr{P}(Y)$ by $(f \times g)(A \times B) = f(A) \times g(B)$, for all $A \times B \in \mathscr{P}(X) \times \mathscr{P}(Y)$.

Theorem 12 *Let $f \sim^X g$ and $h \sim^Y k$. Then, for $A \subseteq X$ and $B \subseteq Y$, $(f \times h)(A \times B) \subset X \times Y \setminus (g \times k)((X \setminus A) \times (Y \setminus B))$.*

Proof Given that $A \times B \in \mathscr{P}(X) \times \mathscr{P}(Y)$. Then, $(f \times h)(A \times B) = f(A) \times h(B) = [X \setminus g(X \setminus A)] \times [Y \setminus k(Y \setminus B)] \subset (X \times Y) \setminus g(X \setminus A) \times k(Y \setminus B) = (X \times Y) \setminus (g \times k)((X \setminus A) \times (Y \setminus B))$. □

Corollary 4 *Let $f \sim^X g$. Then, for $A,\ B \in \mathscr{P}(X)$, $(f \times g)(A \times B) \subset (X \times X) \setminus (g \times f)((X \setminus A) \times (X \setminus B))$.*

Corollary 5 *Let $f \sim^X g$. Then, for $A,\ B \in \mathscr{P}(X)$, $(f \times f)(A \times B) \subset (X \times X) \setminus (g \times g)((X \setminus A) \times (X \setminus B))$.*

3 Operators Due to the Associated Functions

In this section, we define two operators based on associated functions and discuss several properties.

Definition 8 Let $f : \mathscr{P}(X) \to \mathscr{P}(X)$. Then, we define two operators in the following ways:

(a) for a subset $A \in \mathscr{P}(X)$, $f^+(A) = f(A) \setminus A$;
(b) for a subset $A \in \mathscr{P}(X)$, $f^-(A) = A \setminus f(A)$.

Theorem 13 *Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be set-valued set functions such that $f \sim^X g$. Then, the following conditions hold:*

(1) $f^+(\emptyset) = \emptyset$ *and* $f^-(X) = \emptyset$,
(2) $A \cap f^+(A) = \emptyset$ *and* $f(A) \cap f^-(A) = \emptyset$,
(3) $f^-(A) \subseteq A$ *and* $f^+(A) \subseteq f(A)$,
(4) $f^+(A \cup B) = (f^+(A) \setminus B) \cup (f^+(B) \setminus A)$,
(5) $f^-(A \cap B) = (f^-(A) \cap B) \cup (f^-(B) \cap A)$,
(6) *if f is idempotent, then* $f^{++}(A) \subseteq A$,
(7) $f^{--}(A) \subseteq A$.

Proof Conditions (1), (2) and (3) can be obtained easily.

(4) $f^+(A \cup B) = f(A \cup B) \setminus (A \cup B)$. Similarly, we can prove (5).

(6) $f^{++}(A) = f(f^+(A)) \setminus f^+(A) = f(f(A) \setminus A) \setminus (f(A) \setminus A) \subseteq f(f(A)) \setminus (f(A) \setminus A) \subseteq f(A) \setminus (f(A) \setminus A) \subseteq A$. □

Theorem 14 *Let X be a set and $f \sim^X g$. Then,*

(1) *if $f^+(A) = \emptyset$, then $g^-(X \setminus A) = \emptyset \; \forall A \in \mathscr{P}(X)$,*
(2) *if $g^-(A) = \emptyset$, then $f^+(X \setminus A) = \emptyset \; \forall A \in \mathscr{P}(X)$.*

Theorem 15 *Let X be a set, $f \sim^X g$ and $f(A) = X \; \forall A \in \mathscr{P}(X)$. Then, $g^+(A) = \emptyset$ and $g^-(A) = A$.*

Theorem 16 *Let X be a set, $f \sim^X g$ and $A \in \mathscr{P}(X)$. Then, $f^+(X \setminus A) = g^-(A)$.*

Theorem 17 *Let X be a set, $f \sim^X g$, $A \in \mathscr{P}(X)$ and $g(A) \subseteq A \subseteq f(A)$. Then,*

(1) $g(A) = A \setminus F(A)$,
(2) $f(A) = A \cup F(A)$,
(3) $F(A \cup B) \subseteq F(A) \cup F(B)$, *where* $F(A) = f(A) \cap f(X \setminus A)$.

Theorem 18 *Let $f : \mathscr{P}(X) \to \mathscr{P}(X)$ be a set-valued set function such that $f \sim^X f$. Then, $f^+(X \setminus A) = f^-(A)$, for any $A \in \mathscr{P}(X)$.*

Remark 6 If f is additive, i.e., $f(A \cup B) = f(A) \cup f(B)$ and $f \sim^X g$, then $g(A \cap B) = g(A) \cap g(B)$. But, in general, if f is additive set-valued set function, then g is not a additive set-valued set function. We discuss it in the following example:

Example 4 It is easy to check that $cl \sim^X int$. Now, $cl(A \cup B) = cl(A) \cup cl(B)$ but $int(A \cup B) \neq int(A) \cup int(B)$.

We present the necessary and the sufficient conditions for the validity, when f is additive.

(1) $f(A \cap B) = f(A) \cap f(B)$,
(2) $g(A \cup B) = g(A) \cup g(B)$.

Theorem 19 *Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions such that $f \sim^X g$ and for $A, B \in \mathscr{P}(X)$ with f is additive. Then, the following statements are equivalent:*

(1) $g(A \cup B) = g(A) \cup g(B)$ *(additive),*
(2) $F(A) \setminus g(B) = f(A) \setminus g(A \cup B)$ *and* $F(B) \setminus g(A) = f(B) \setminus g(A \cup B)$,
(3) $F(A \cup B) = [F(A) \setminus g(B)] \cup [F(B) \setminus g(A)]$.

Proof (1) $\Rightarrow$ (2): We know that

$$F(A) \setminus g(B) = [f(A) \setminus g(A)] \setminus g(B) = f(A) \setminus [g(A) \cup g(B)]$$
$$\text{and F(B)} \setminus \text{g(A)} = [f(B) \setminus g(B)] \setminus g(A) = f(B) \setminus [g(A) \cup g(B)]. \quad (1)$$

Hence, by (1), we have $F(A) \setminus g(B) = f(A) \setminus g(A \cup B)$ and $F(B) \setminus g(A) = f(B) \setminus g(A \cup B)$.

(2) $\Rightarrow$ (3): If (2) holds, then it follows that

$$\begin{aligned} F(A \cup B) &= f(A \cup B) \cap f(X \setminus (A \cup B)) \\ &= f(A \cup B) \cap [X \setminus g(X \setminus (A \cup B)) \\ &= f(A \cup B) \setminus g(A \cup B) \\ &= (f(A) \cup f(B)) \setminus g(A \cup B) = (f(A) \setminus g(A \cup B)) \cup (f(B) \setminus g(A \cup B)) \\ &= [F(A) \setminus g(B)] \cup [F(B) \setminus g(A)]. \end{aligned}$$

(3) $\Rightarrow$ (1): By Eq. (1), we get

$$\begin{aligned} [F(A) \setminus g(B)] \cup [F(B) \setminus g(A)] &= [f(A) \setminus [g(A) \cup g(B)] \cup [f(B) \setminus [g(A) \cup g(B)] \\ &= [f(A) \cup f(B)] \setminus [g(A) \cup g(B)] \\ &= [f(A \cup B)] \setminus [g(A) \cup g(B)] \end{aligned}$$

Now, by (3), we have $F(A \cup B) = [f(A \cup B)] \setminus [g(A) \cup g(B)]$. Thus,

$$g(A \cup B) = f(A \cup B) \setminus F(A \cup B) = f(A \cup B) \setminus [f(A \cup B)] \setminus [g(A) \cup g(B)] = g(A) \cup g(B).$$

Hence, the proof is done. □

Theorem 20 *Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions such that $f \sim^X g$ with f is additive. Then, for $A, B \in \mathscr{P}(X)$, $g(A \setminus B) = g(A) \setminus f(B)$. But, in general $f(A \setminus B) \neq f(A) \setminus g(B)$. Also, the following statements are equivalent:*

(1) $f(A \setminus B) = f(A) \setminus g(B)$,
(2) $g[(X \setminus A) \cup B] = g(X \setminus A) \cup g(B)$,
(3) $f[A \cap (X \setminus B)] = f(A) \cap f(X \setminus B)$,
(4) $F(A) \setminus g(B) = f(X \setminus A) \setminus g((X \setminus A) \cup B)$ *and* $f(A) \cap F(B) = f(B) \setminus g((X \setminus A) \cup B)$.

Proof It is clear that

$$g(A) \setminus f(B) = g(A) \cap [X \setminus f(B)] = g(A) \cap g[X \setminus B] = g(A \cap (X \setminus B)) = g(A \setminus B).$$

(1)$\Rightarrow$ (2): If $f(A \setminus B) = f(A) \setminus g(B)$, then

$$\begin{aligned} g(X \setminus A) \cup g(B) &= (X \setminus f(A)) \cup g(B) = X \setminus [f(A) \cap (X \setminus g(B))] = X \setminus (f(A) \setminus g(B)) \\ &= X \setminus f(A \setminus B) = X \setminus f(A \cap (X \setminus B)) = g(X \setminus (A \cap (X \setminus B))) \\ &= g((X \setminus A) \cup B). \end{aligned}$$

(2) $\Rightarrow$ (3): If $g[(X \setminus A) \cup B] = g(X \setminus A) \cup g(B)$, then

$$\begin{aligned} f(A) \cap f(X \setminus B) &= f(X \setminus (X \setminus A)) \cap f(X \setminus B) = [X \setminus g(X \setminus A)] \cap [X \setminus g(B)] \\ &= X \setminus [g(X \setminus A) \cup g(B)] = X \setminus g[(X \setminus A) \cup B] = f[X \setminus ((X \setminus A) \cup B)] \\ &= f[(X \setminus X \setminus A) \cap (X \setminus B)] = f[A \cap (X \setminus B)]. \end{aligned}$$

(3) $\Rightarrow$ (1): If $f[A \cap (X \setminus B)] = f(A) \cap f(X \setminus B)$, then

$$f(A \setminus B) = f(A \cap (X \setminus B)) = f(A) \cap f(X \setminus B) = f(A) \cap (X \setminus g(B)) = f(A) \setminus g(B).$$

(2) $\Leftrightarrow$ (4): We know that

$$f(A) \cap F(B) = F(B) \cap [X \setminus X \setminus f(A)] = F(B) \cap (X \setminus g(X \setminus A)) = F(B) \setminus g(X \setminus A).$$

It is obvious that $F(A) = F(X \setminus A)$. Therefore, (4) is equivalent to

$$F(X \setminus A) \setminus g(B) = f(X \setminus A) \setminus g((X \setminus A) \cup B) \text{ and}$$

$$F(B) \cap f(A) = F(B) \setminus g(X \setminus A) = f(B) \setminus g((X \setminus A) \cup B).$$

By substituting $X \setminus A$ for A in (1) and (2) of Theorem 19, we conclude that the above equality is equivalent to (2). Hence, we proved (2) $\Leftrightarrow$ (4). □

Theorem 21 *Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions such that $f \sim^X g$ with f is additive and $g(H) \subseteq H \subseteq f(H)$, for all $H \in \mathscr{P}(X)$. Then, for $A, B \in \mathscr{P}(X)$, the following statements are equivalent:*

(1) $f(A \cap B) = f(A) \cap f(B)$,
(2) $f(A) \cap F(B) = f(A \cap B) \setminus g(B)$ *and* $f(B) \cap F(A) = f(A \cap B) \setminus g(A)$,
(3) $F(A \cap B) = [F(A) \cup F(B)] \cap [f(A) \cap f(B)]$.

Proof We can check that

$$\begin{aligned}
[f(A) \cap f(B)] \setminus g(A \cap B) &= [f(A) \cap f(B)] \setminus [g(A) \cap g(B)] \\
&= [(f(A) \cap f(B)) \setminus g(B)] \cup [(f(A) \cap f(B)) \setminus g(A)] \\
&= [f(A) \cap f(B) \cap (X \setminus g(B))] \cup [f(A) \cap f(B) \cap (X \setminus g(A))] \\
&= [f(A) \cap F(B)] \cup [f(B) \cap F(A)]. \qquad (2)
\end{aligned}$$

Also, we can find that

$$\begin{aligned}
[F(A) \cup F(B)] \cap [f(A) \cap f(B)] &= [F(B) \cap f(A) \cap f(B)] \cup [F(A) \cap f(A) \cap f(B)] \\
&= [f(A) \cap F(B)] \cup [f(B) \cap F(A)]. \qquad (3)
\end{aligned}$$

(1) $\Rightarrow$ (2): If $f(A \cap B) = f(A) \cap f(B)$, then

$$f(A \cap B) \setminus g(B) = [f(A) \cap f(B)] \setminus g(B) = f(A) \cap f(B) \cap (X \setminus g(B)) = f(A) \cap F(B).$$

Similarly, we prove that $f(B) \cap F(A) = f(A \cap B) \setminus g(A)$.

(2) $\Rightarrow$ (3): If $f(A) \cap F(B) = f(A \cap B) \setminus g(B)$ and $f(B) \cap F(A) = f(A \cap B) \setminus g(A)$ hold, then it follows from (2) and Eq. (3) that

$$\begin{aligned}[F(A) \cup F(B)] \cap [f(A) \cap f(B)] &= [f(A) \cap F(B)] \cup [f(B) \cap F(A)] \\ &= [f(A \cap B) \setminus g(B)] \cup [f(A \cap B) \setminus g(A)] \\ &= f(A \cap B) \setminus [g(A) \cap g(B)] \\ &= f(A \cap B) \setminus g(A \cap B) = F(A \cap B).\end{aligned}$$

(3) $\Rightarrow$ (1): If $F(A \cap B) = [F(A) \cup F(B)] \cap [f(A) \cap f(B)]$ holds, then combining (3), Eqs. (2) and (3), we get the equality $F(A \cap B) = [f(A) \cap f(B)] \setminus g(A \cap B)$. Hence, we conclude that

$$f(A \cap B) = g(A \cap B) \cup F(A \cap B) = g(A \cap B) \cup [(f(A) \cap f(B)) \setminus g(A \cap B)] = f(A) \cap f(B).$$

□

Theorem 22 *Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions such that $f \sim^X g$ with f is additive. If $g(A \cup B) = g(A) \cup g(B)$ for $A, B \in \mathscr{P}(X)$, then $F(A) \cap F(B) \subseteq F(A \cup B)$.*

Proof

$$\begin{aligned}F(A) \cap F(B) &= [f(A) \cap (X \setminus g(A))] \cap [f(B) \cap (X \setminus g(B))] \\ &= f(A) \cap f(B) \cap (X \setminus g(A) \cap (X \setminus g(B) \\ &= f(A) \cap f(B) \cap (X \setminus (g(A) \cup g(B))) \\ &= [f(A) \cap f(B)] \setminus [g(A) \cup g(B)],\end{aligned}$$

and $F(A \cup B) = f(A \cup B) \setminus g(A \cup B) = [f(A) \cup f(B)] \setminus g(A \cup B)$. Thus,

$$[f(A) \cap f(B)] \setminus [g(A) \cup g(B)] \subseteq [f(A) \cup f(B)] \setminus g(A \cup B).$$

It is obvious that $f(A) \cap f(B) \subseteq f(A) \cup f(B)$. Hence, the theorem is proven. □

Lemma 1 *Let $f, g : \mathscr{P}(X) \to \mathscr{P}(X)$ be two set-valued set functions such that $f \sim^X g$ with f is additive. Then, for $A, B \in \mathscr{P}(X)$, the following statements are equivalent:*

(1) $f(A \cap B) = f(A) \cap f(B)$,
(2) $g[(X \setminus A) \cup (X \setminus B)] = g(X \setminus A) \cup g(X \setminus B)$.

Proof By substituting $X \setminus B$ for B in (1) and (2) of Theorem 20, we can prove the equivalence of both conditions (1) and (2). □

4 Filter Convergent Due to the Associated Functions

In this section, we establish some relations between associated functions and filter.

Definition 9 ([4]) Let X be a set. A collection $\mathscr{F}$ of subsets of X is said to be a *filter on* X if

(a) $\emptyset \notin \mathscr{F}$;
(b) for $A \supseteq B \in \mathscr{F}$, $A \in \mathscr{F}$;
(c) for $A,\ B \in \mathscr{F}$, $A \cap B \in \mathscr{F}$.

Definition 10 Let $f \sim^X g$. A subset A of X is said to be $\sim^X$ *open* if $A = f(A)$. It is obvious that A is $\sim^X$ open if and only if $g(X \setminus A) = X \setminus A$.

Let $x \in X$. Then, the collection of all $\sim^X$ open sets containing x is denoted as $\sim (x)$, i.e., $\sim (x) = \{A \in \mathscr{P}(X) \mid x \in A,\ A = f(A)\}$.

Let $f \sim^X g$ and $h \sim^Y k$. If A is $\sim^X$ open and B is $\sim^Y$ open, then it is obvious that $A \times B = f(A) \times h(B) = (f \times h)(A \times B)$. Thus, we have $A \times B$ is $\sim^{X \times Y}$ open in $X \times Y$.

Now, we define the convergence of a filter in terms of $\sim^X$ open sets.

Definition 11 Let $f \sim^X g$ and $\mathscr{F}$ be a filter on X. Then, $\mathscr{F}$ is said to be $\sim^X$ *converges to* $x \in X$ if $\sim (x) \subseteq \mathscr{F}$.

Suppose X be a topological space. Consider 'cl' and 'int' as the closure and interior operators, respectively. If we consider $cl \sim^X int$, then the ordinary convergence and $\sim^X$ convergence of a filter are synonymous. Following is the difference between them:

Example 5 Consider an ideal topological space $(X, \tau, \mathscr{I})$, where $X = \{x, y, z\}$, $\tau = \{\emptyset, X, \{x, y\}\}$ and $\mathscr{I} = \{\emptyset, \{x\}\}$. Then, we have $cl \sim^X int$ and $()^* \sim^X \psi$. Then, the filter $\mathscr{F} = \{X, \{x, y\}\}$ converges to the points x, y and z. Further, for $\{z\}$, $\{z\}^* = \{z\}$ and $\{z\} \in \sim (z)$. Here, $\{z\} \notin \mathscr{F}$. Thus, $\mathscr{F}$ is not $\sim^X$ convergent to z.

Example 6 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then, $\{b\}^* = \emptyset$, $\{c\}^* = \{b, c\}$, $\{a, c\}^* = \{a, b, c\}$, $\{a, b\}^* = \{a, b, c\}$, $\{a\}^* = \{a, b, c\}$ and $X^* = X$. Then, the filter $\mathscr{F} = \{X, \{b, c\}\}$, $\sim^X$ converges to a but it is not convergent to a.

Theorem 23 *Let $f \sim^X g$, x and y be two distinct points of X. If $A \cap B = \emptyset$ for all $A \in \sim (x)$ and $B \in \sim (y)$, then a filter $\mathscr{F}$ on X is not $\sim^X$ convergent to the points x and y simultaneously.*

Proof If possible, suppose that $\mathscr{F} \sim^X$ converges to x as well as y. Then, $\sim (x) \subseteq \mathscr{F}$ and $\sim (y) \subseteq \mathscr{F}$. Thus, for all $C \in \sim (x)$ and for all $D \in \sim (y)$, $C \cap D \in \mathscr{F}$. It yields $\emptyset \in \mathscr{F}$. But, this is a contradiction. Hence, the proof is done. □

Theorem 24 *Let $f \sim^X g$ and $A \subseteq X$. If $\mathscr{F}$ be a filter on X containing A and $\mathscr{F}$ $\sim^X$ converges to x, then $A \cap U \neq \emptyset$, for all $U \in \sim (x)$.*

Proof Given that $\sim (x) \subseteq \mathscr{F}$ and $A \in \mathscr{F}$. Thus, from the definition of filter, we have $U \cap A \neq \emptyset$, for all $U \in \sim (x)$. □

Converse of the above theorem is not true in general. We have the following example:

Example 7 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, c\}, X\}$. We know that $cl \sim^X int$. Consider the filter $\mathscr{F} = \{\{b, c\}, X\}$ on X. Then, for the point c, $U \cap A \neq \emptyset$ for all $U \in\sim (c)$ and for all $A \in \mathscr{F}$. But, $\mathscr{F}$ is not $\sim^X$ convergent to c.

Theorem 25 *Let $f \sim^X g$ and $f(C \cap D) = f(C) \cap f(D)$, for all $C, D \in \mathscr{P}(X)$. If $A \cap U \neq \emptyset$ for all $U \in\sim (x)$, then there exists a filter $\mathscr{F}$ on X such that $A \in \mathscr{F}$ and $\mathscr{F} \sim^X$ converges to x.*

Proof Consider the filter $\mathscr{F} = \{B \supseteq A \cap U \mid U \in\sim (x)\}$ on X. It is easily shown that $A \in \mathscr{F}$ and $\mathscr{F} \sim^X$ converges to x. □

Definition 12 ([21]) A filter $\mathscr{U}$ on a set X is said to be an ultrafilter if it is a maximal element in the collection of all filters on X, partially ordered by set inclusion, that is not properly contained in any filter on X.

Proposition 1 [21] *Let X be a non-empty set. A filter $\mathscr{U}$ on X is an ultrafilter if and only if, for any $A \subseteq X$, either $A \in \mathscr{U}$ or $X \setminus A \in \mathscr{U}$.*

Theorem 26 *Let $f \sim^X g$. Let $\mathscr{U}$ be an ultrafilter on X. If $U \cap A \neq \emptyset$ for all $U \in\sim (x)$ and for all $A \in \mathscr{U}$, then $\mathscr{U} \sim^X$ converges to x.*

Proof If possible, suppose that $\mathscr{U}$ is not $\sim^X$ convergent to x. Then, there exists $U \in\sim (x)$ such that $U \notin \mathscr{U}$. Then, from above characterization, $X \setminus U \in \mathscr{U}$. By the given condition, $(X \setminus U) \cap U \neq \emptyset$, a contradiction. Thus, $\mathscr{U} \sim^X$ converges to x. □

Theorem 27 ([4]) *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two filters on the sets X_1 and X_2, respectively. Then, $\mathscr{F}_1 \times \mathscr{F}_2 = \{A \times B \mid A \in \mathscr{F}_1$ and $B \in \mathscr{F}_2\}$ is a filter base for a filter on $X_1 \times X_2$.*

Theorem 28 *Let $f \sim^X g$ and $h \sim^Y k$. If a filter $\mathscr{F}_X$ on X, $\sim^X$ converges to x and a filter $\mathscr{F}_Y$ on Y, $\sim^Y$ converges to y, then the filter $\mathscr{F}$ on $X \times Y$ generated by the filter base $\mathscr{F}_X \times \mathscr{F}_Y$, $\sim^{X\times Y}$ converges to (x, y).*

Proof Let $A \times B$ be a $\sim^{X\times Y}$ open set containing (x, y) in $X \times Y$. Then, $A \times B = (f \times h)(A \times B)$, and hence $A = f(A)$ and $B = g(B)$. Thus, A is a $\sim^X$ open set containing x and B is a $\sim^Y$ open set containing y. Since, $\mathscr{F}_X \sim^X$ converges to x, then $A \in \mathscr{F}_X$. Similarly, $B \in \mathscr{F}_Y$. So, $A \times B \in \mathscr{F}$. Thus, the filter $\mathscr{F}$ is $\sim^{X\times Y}$ convergent on $X \times Y$ and $\sim^{X\times Y}$ converges to (x, y). □

5 Hypergraph Structures with Associated Functions

Hypergraphs are highly used in various social networks [2, 20, 23, 24]. Their various topological and network based properties are studied in interdisciplinary manners. In this section, we connect associated functions with hypergraph structures by introducing two types of hypergraphs. Here, we do not study topological or network based properties, but we show that associated functions can be connected to hypergraph structures.

Definition 13 Let V be a non-empty finite set of points. Define a set-valued set function $g : \mathscr{P}(V) \to \mathscr{P}(V)$. Then, the pair (V, E_g) is called *a set-valued set function graph of first kind* (denoted as $SVSFG_g$) generated by g, where $E_g = \{g(A_i)|A_i \in \mathscr{P}(V); i \in \Delta\}$ and Δ is a finite index set. The sets $g(A_i)$ are called set-valued edges and the points of V are called vertices.

Definition 14 Let V be a non-empty finite set of vertices and (V, E_g) be a $SVSFG_g$. Define another function $f : \mathscr{P}(V) \to \mathscr{P}(V)$ be set-valued set function such that $f \sim^X g$, then the pair $(V, {}_gE_f)$ is said to be *associated function graph of f with respect to g* (it is denoted as ${}_gAFG_f$); where ${}_gE_f = \{f(A_i)|A_i \in \mathscr{P}(V); i \in \Delta, f \sim^X g\}$ and Δ is a finite index set. The elements of ${}_gE_f$ are called AFG-edges.

The following definition is the generalization of Sampathkumar and Pushpalatha [17]:

Definition 15 Let V be a non-empty finite set of points. Define a set-valued set function $g : \mathscr{P}(V) \to \mathscr{P}(V)$ such that

(a) $|g(A_i)| \geq 2$, for each $g(A_i) \in E_g^s$;

(b) $|g(A_i) \cap g(A_j)| \leq \epsilon$, for all $A_i, A_j \in \mathscr{P}(V), i \neq j$ where ϵ is a natural number;

(c) $\bigcup_{i=1}^{n} g(A_i) = V$.

Then, the triplet (V, E_g^s, ϵ) is said to be set-valued set function graph of second kind generated by g (denoted as $SVSFG_g^s$), where $E_g^s = \{g(A_1), g(A_2), \ldots, g(A_n)\}$. The elements of E_g^s are called S-edges.

Definition 16 Let (V, E_g^s, ϵ) be a $SVSFG_g^s$. Define another function $f : \mathscr{P}(V) \to \mathscr{P}(V)$ be set-valued set function such that $f \sim^X g$. Then, the triplet $(V, {}_gE_f^s, \epsilon)$ is said to be *S-associated function graph* $({}_gSAFG_f)$ of f with respect to g, where ${}_gE_f^s = \{f(A_1), f(A_2), \ldots, f(A_n)|f \sim^X g, A_i \in \mathscr{P}(V); i \in \Delta\}$ and Δ is a finite index set. The elements of ${}_gE_f^s$ are called SAF-edges.

Theorem 29 *Let V be a non-empty finite set of vertices and (V, E_g) be a $SVSFG_g$. Then, (V, E_g) is a hypergraph.*

Theorem 30 *Let V be a non-empty finite set of vertices and (V, E_g) be a $SVSFG_g$. Then, $(V, {}_gE_f)$ is a hypergraph.*

Proof Given V be a non-empty finite set of vertices and (V, E_g) be a $SVSFG_g$. Then, $g : \mathscr{P}(V) \to \mathscr{P}(V)$ is a set-sevalued set function and $g(B) \neq V$ for all $B \subseteq V$. We define another set-sevalued set function $f : \mathscr{P}(V) \to \mathscr{P}(V)$ such that $f \sim^X g$. Thus, $f(A) = V \setminus g(V \setminus A) \neq \emptyset$. Let ${}_gE_f = \{f(A) : A \subseteq V \text{ and } f \sim^X g\}$. Thus, $(V, {}_gE_f)$ is a hypergraph. □

Theorem 31 *Let V be a non-empty finite set of points and (V, E_g^s, ϵ) is a $SVSFG_g^s$. Then, (V, E_g^s, ϵ) is a hypergraph.*

Definition 17 Let $g : \mathscr{P}(V) \to \mathscr{P}(V)$ be a set-valued set function. Then, g is said to satisfy *Complement Intersection Property (CIP)* if $\bigcap_{i=1}^{n} g(V \setminus A_i) = \emptyset$, for all $A_i \in \mathscr{P}(V)$.

Theorem 32 *Let V be a non-empty finite set of vertices, g satisfies CIP, (V, E_g^s, ϵ) is a $SVSFG_g^s$, $2 \leq |g(A_i)| \leq |V| - 2$ and $|V| > \epsilon$. Then, $(V, {}_gE_f^s, \delta)$ is a hypergraph, where $\delta = |V| - \epsilon$, $|V| \geq 4$.*

Proof Let V be a non-empty finite set of vertices and (V, E_g^s, ϵ) is a $SVSFG_g^s$, where $|V| \geq 4$ and $|V| > \epsilon$. Then, the set-valued function $g : \mathscr{P}(V) \to \mathscr{P}(V)$ satisfies the conditions (1), (2), (3) of Definition 15. We define $f : \mathscr{P}(V) \to \mathscr{P}(V)$ such that $f \sim^V g$. Then,

(1) $|f(A_i)| = |V \setminus g(V \setminus A_i)| \geq 2$,

(2) $|f(A_i) \cap f(A_j)| = |\{V \setminus g(V \setminus A_i)\} \cap \{V \setminus g(V \setminus A_j)\}| = |V \setminus \{g(V \setminus A_i) \cup g(V \setminus A_j)\}| \leq \delta$, where $\delta = |V| - \epsilon$,

(3) $\bigcup_{i=1}^{n} f(A_i) = \bigcup_{i=1}^{n} \{V \setminus g(V \setminus A_i)\} = V \setminus \bigcap_{i=1}^{n} g(V \setminus A_i) = V$.

Thus, $(V, {}_gE_f^s, \delta)$ is a hypergraph. □

6 Conclusion

In this chapter, we introduced associated functions and discussed their several properties. The importance of associated functions lie on the ideas of set-valued set functions. The results showed possible connections of semigroup theory with topology via associated functions. Moreover, we showed that the associated functions generate hypergraphs. For these purposes, we introduced two types of hypergraphs and discussed a few fundamental properties. We opened the way to connect hypergraph theory with topology via associated functions. Thus, this chapter may open scopes of several new interdisciplinary ideas in computer science, biology, sociology, engineering, social networks, etc.

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Hunting Cooperation and Fear Effect in Parallelism with Immature's Hunting in a Predator-Prey Symbiosis

Tapasvini Roy and Debasish Bhattacharjee

Abstract A detailed anatomy of the synergy between a predator and its prey is done, whereupon both the immature and matured stages of the predator are endowed with the potency to predate, in tandem to which, matured predators' hunting cooperation and impact of fear are incorporated. In addition, the contexture of role reversal between the immature predator and the prey is envisaged here. The model thus contrived manifests three types of biologically viable equilibrium points, all of whose stability analyses are executed, also the existence of limit cycles are authenticated through both theoretical and graphical methods. The rich dynamical scenarios of the model are visualized through numerical simulations, thereby establishing the eminence of some vital parameters of the model.

1 Introduction

The dynamic symbiosis between the predator and its prey has long been one of the paramount importance in the scientific community due to its universal applicability. Since the pioneering work of Lotka and Volterra, the representation of the afore mentioned symbiosis through mathematical models [1, 7, 8, 17–20, 36, 39] has gained much prominence owing to the enormous aspects of the relationship, which makes it both interesting and arduous to form.

Cooperative hunting is a permeating phenomenon among the predators, where the predators seize their prey together as a group. Gray wolves, lions, jackals, cheetahs, Harris's hawks, army ants, and spiders are only a fistful among the immensurate amount of species that display cooperative hunting while predating [2–4]. In recent times, a liberal amount of authors have probed into the impact of hunting cooperation on the predator-prey symbiosis [5–10].

The reverberation of predation on the prey species, apart from being lethal, is also non-lethal. Ascribable to fear of predation risk, the prey population alters their

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pasturage to a safer place from their lucrative pasturage, regulates their reproductive chronology and so forth, thereby effectuating changes in the composition of their population, which is to say that these types of anti-predator behaviors are poised in reducing the reproduction as a long-term cost [11–14]. The consequence of fear effect in the predator-prey symbiosis too has been explored through mathematical modeling [15–18].

Lionesses are known to hunt cooperatively [34], while at the same time they incite the anti-predatory behavior in their prey owing to its fear [35]. Thus, with this incentive from the literature survey, cooperative hunting and impact of fear are combined in this paper. A structure where the combined effect of hunting cooperation and the consequence of fear effect in the prey has been proposed through mathematical modeling in [7, 19, 20].

During an interaction, the warring species are categorized as either the prey or the predator, but on account of several documentations the roles of both species are sometimes known to be reversed, for example, lobsters and whelks [21], lizards and scorpions [22].

Furthermore, it has been demonstrated that the prey are particularly able to kill the immature predators rather than the adult ones, with the reason of the adults being experienced in handling the prey [23]. Also, several species of immature predators are described to possess the ability to predate [24–27]. Thus, a quite logical scenario is envisioned, whereupon the immature predators are at mortal peril while hunting their prey owing to the dangerousness of the prey. The scenario between bats and spiders [29], pumas and ungulates [28], snakes and spiders [30] can be taken as examples for such type of interaction. The mathematical analysis of this type of interaction between the prey and predator species is crucial because hunting by immature predators and its adverse consequences cause immense changes in the biosphere.

Kaushik and Banerjee [37, 38] investigate this particular scenario, where immature predators hunt and get killed as a consequence of their naivety toward prey's anti-predatory behavior, through stage structured mathematical model with and without delay respectively, with the help of Holling type 1 functional response. Ghanbari [36] investigates the same as [38] except with the help of Caputo-Fabrizio derivative. Bhattacharjee et al. [39] study the scenario, where immature predators hunt and get killed as a consequence of both prey's anti-predatory behaviour and habitat complexity, with the help of ratio-dependent functional response.

The pivotal essence of this paper is to construct a mathematical model where the immature stage and matured stage are the two stages in which the predator species is structured, and both the stages are endowed with the potentiality to predate. The scenario of cooperative hunting is envisaged in the adult predators only, as the immature predators, being inexperienced and lacking robustness, are not fully efficient in handling the prey, and therefore are unable to work collectively. For the same reason, the impact of fear is caused by the matured predator only. Furthermore, the inefficiency of the immature predator can cause their death during the predation process. Therefore, the use of both lethal and non-lethal anti-predator behavior of the prey against the predator species is probed into. One such biological scenario can be given between the wolves and their prey, elks or white-tailed deer. The role reversal scenario between

the immature wolves and elks can be envisioned as given in [39]. Also, wolves are known to hunt cooperatively and at the same time elks and white-tailed deer respond in the presence of wolves by altering patterns of aggregation, habitat selection, vigilance, foraging, and sensitivity to environmental conditions [31–33, 40, 41]. The combination of cooperative hunting and fear effect with negative repercussion of predation in the immature predator has not yet been reckoned with, thus ascertaining this paper to be one of a kind.

In Sect. 2 the model is formulated, and its boundedness is proven in Sect. 3. The equilibrium points and their stabilities are discussed in Sect. 4, and the existence of limit cycle is proven analytically in the next section. In Sect. 6, the numerical simulation is presented, which is followed by conclusion in Sect. 7.

2 Formulation of the Model

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k}) - a_1xy - a_2xz,\\
\frac{dy}{dt} &= ua_2xz - bxy - cy - d_1y,\\
\frac{dz}{dt} &= cy - d_2z,\\
&\text{with initial conditions } x(0) > 0,\ y(0) > 0,\ z(0) > 0.
\end{aligned} \tag{1}$$

Owing to limited natural resources, in the absence of predators, the prey population x is assumed to have a logistic growth rate. The predators are partitioned into two stages—y is the immature predators' population and z is the matured predators' population. Both the stages of predator follow Holling type 1 functional response while predating, a_1 is the predation rate of immature predators and a_2 is the predation rate of matured predators. The death rate of immature predators due to their inefficiency while predating is b, also their natural death rate is taken as d_1. u is the conversion rate of killed prey into immature predators. c is the transition rate of immature predators into matured stage. d_2 is the matured predators' natural death rate. The model 1 is emanated from [38].

Now, for the incorporation of cooperative hunting, h, the predation rate of matured predator is remolded from a_2 to $(a_2 + hz)$. Also at the same time the incorporation of impact of fear, s necessitates alteration of the growth rate of prey from $rx(1 - x/k)$ to $\frac{rx(1-x/k)}{1+hsz}$, as per the fact that the fear among the prey may be because of cooperative hunting in the matured predators.
Thus, our model is

$$\begin{aligned}
\frac{dx}{dt} &= \frac{rx(1-x/k)}{1+hsz} - a_1xy - (a_2+hz)xz, \\
\frac{dy}{dt} &= u(a_2+hz)xz - bxy - cy - d_1y, \\
\frac{dz}{dt} &= cy - d_2z, \\
&\text{with initial conditions } x(0) > 0,\ y(0) > 0,\ z(0) > 0.
\end{aligned} \tag{2}$$

3 Boundedness

Consider a variable B such as $B = ux + y + z$.

$$\begin{aligned}
B &= ux + y + z, \\
i.e.\ B' &= ux' + y' + z' \text{ [differentiating with respect to time, t]}, \\
i.e.\ B' &= \frac{urx(1-x/k)}{1+hsz} - ua_1xy - bxy - d_1y - d_2z \\
&\leq urx(1-\frac{x}{k}) - d_1y - d_2z,
\end{aligned}$$

$$\begin{aligned}
i.e.,\ B' + \nu B &\leq urx(1-x/k) + u\nu x - (d_1-\nu)y - (d_2-\nu)z \text{ [where } \nu = min\{d_1, d_2\}] \\
&\leq ux(r - \frac{r}{k}x + \nu) \quad = -\frac{ur}{k}x^2 + u(r+\nu)x \\
&= -\frac{ur}{k}\left(x^2 - \frac{2kr(r+\nu)x}{2r} + \frac{k^2(r+\nu)^2}{4r^2}\right) + \frac{k^2(r+\nu)^2}{4r^2}\frac{ur}{k} \\
&\leq \frac{uk}{4r}(r+\nu)^2. \\
\therefore\ B &\leq \frac{uk(r+\nu)^2}{4r\nu}\left(1-e^{\nu t}\right) + B(x(0), y(0), z(0))e^{-\nu t}.
\end{aligned}$$

As $t \to \infty$, we get, $\limsup_{t\to\infty} B \leq \frac{uk(r+\nu)^2}{4r\nu}$.
Therefore, the solutions of system of Eq. 2 cannot increase unrestrainedly.

4 Equilibrium Points and Their Stabilities

There exist three viable equilibrium points for the system of Eq. 2:

1. The extinction equilibrium (0, 0, 0),
2. The predator free equilibrium $(k, 0, 0)$, and

3. The co-existing equilibrium (x^*, y^*, z^*),
where $x^* = \frac{k}{r}(r - (1 + hsz^*)(a_1y^* + z^*(a_2 + hz^*)))$, $y^* = \frac{d_2(d_2(d_1+bx^*)+c(d_2-a_2ux^*))}{c^2hux^*}$, $z^* = \frac{d_2(d_1+bx^*)+c(d_2-a_2ux^*)}{chux^*}$.

In this section, we endeavor towards analyzing the local stability analysis of the three equilibrium points by using the technique of linearization. The Jacobian matrix J of the system of Eq. 2 is

$$\begin{pmatrix} \frac{(1-\frac{x}{k})r}{hsz+1} - \frac{xr}{k(hsz+1)} - a_1y - z(a_2 + hz) & -a_1x & -hzx - (a_2 + hz)x - \frac{hrs(1-\frac{x}{k})x}{(hsz+1)^2} \\ uz(a_2 + hz) - by & -c - d_1 - bx & huxz + ux(a_2 + hz) \\ 0 & c & -d_2 \end{pmatrix}. \tag{3}$$

Theorem 1 *The extinction equilibrium point* $(0, 0, 0)$ *is a saddle point.*

Proof At the equilibrium $(0, 0, 0)$, the Jacobian 3 becomes $\begin{pmatrix} r & 0 & 0 \\ 0 & -c - d_1 & 0 \\ 0 & c & -d_2 \end{pmatrix}$,
whose eigenvalues are $-c - d_1, \quad -d_2, \quad r$.
Hence, making the equilibrium point $(0, 0, 0)$ always a saddle point.

Theorem 2 *The predator free equilibrium* $(k, 0, 0)$ *is locally asymptotically stable when* $a_2 < \frac{bd_2k+cd_2+d_1d_2}{cku}$.

Proof $\begin{pmatrix} -r & -a_1k & -a_2k \\ 0 & -c - d_1 - bk & a_2ku \\ 0 & c & -d_2 \end{pmatrix}$ is Jacobian 3 at $(k, 0, 0)$, whose eigenvalues are
$-r$,
$\frac{1}{2}\left(-\sqrt{(bk + c + d_1 + d_2)^2 - 4(c(d_2 - a_2ku) + d_2(bk + d_1))} - bk - c - d_1 - d_2\right)$,
$\frac{1}{2}\left(\sqrt{(bk + c + d_1 + d_2)^2 - 4(c(d_2 - a_2ku) + d_2(bk + d_1))} - bk - c - d_1 - d_2\right)$.
Clearly, only the third eigenvalue can be positive. But it too can be negative, like the first and second eigenvalues, iff

$$a_2 < \frac{bd_2k + cd_2 + d_1d_2}{cku}.$$

Hence, our proof.

Theorem 3 *For* $\mu_1 > 0, \ \mu_3 > 0, \ \mu_1\mu_2 - \mu_3 > 0$, *the values of* μ_i *are written in the proof, the coexisting equilibrium* (x^*, y^*, z^*) *is locally asymptotically stable.*

Proof Putting the equilibrium point (x^*, y^*, z^*) in the Jacobian 3, we would get the required matrix, whose characteristic equation is

$$\lambda^3 + \mu_1\lambda^2 + \mu_2\lambda + \mu_3 = 0, \tag{4}$$

where $\mu_1 = c + d_1 + d_2 + bx^* + a_1y + a_2z^* - hz^{*2} + (r(k - 2x^*))/(k + hksz^*)$,

$$\begin{aligned}
\mu_2 = &-(-cd_2 - d_1d_2 - bd_2x^* + a_2cux^* - a_1cy^* - a_1d_1y^* - a_1d_2y^* - (-2chux^* \\
&+ a_2(c + d_1 + d_2 + bx^* + a_1ux^*))z^* - h(c + d_1 + d_2 + bx^* + a_1ux^*)z^{*2} \\
&+ \frac{r(k - 2x^*)(c + d_1 + d_2 + bx^*)}{k + hksz^*}),
\end{aligned}$$

$$\begin{aligned}
\mu_3 = &-[(1/(k(1 + hsz^*)^2))(-d_2(1 + hsz^*)(d_1(2rx^* + k(-r + (1 + hsz^*)(a_1y^* \\
&+ z^*(a_2 + hz^*)))) + x^*(a_1kuz^*(a_2 + hz^*)(1 + hsz^*) + b(2rx^* + k(-r + z^*(a_2 \\
&+ hz^*)(1 + hsz^*))))) + c(-d_2(1 + hsz^*)(2rx^* + k(-r + (1 + hsz^*)(a_1y^* \\
&+ z^*(a_2 + hz^*)))) + x^*(a_2(rux^*(2 + 3\,hsz^*) + k((b + a_1u)y^*(1 + hsz^*)^2 - \\
&r(u + 2\,hsuz^*))) + h(by^*(-rsx^* + k(rs + 2z^*(1 + hsz^*)^2)) + uz^*(rx^*(4 + 5\,hsz^*) \\
&+ k(2a_1y^*(1 + hsz^*)^2 - r(2 + 3\,hsz^*)))))))].
\end{aligned}$$

Using the Routh-Hurwitz criteria, all of the eigenvalues of 4 would be negative iff

$$\mu_1 > 0,\ \mu_3 > 0,\ \mu_1\mu_2 - \mu_3 > 0.$$

Hence, our proof.

5 Limit Cycle

The birth or death of limit cycle surrounding an equilibrium point is tagged as Hopf bifurcation (also known as Poincare-Andronov-Hopf bifurcation). In this section, the focus is kept on deciphering the requirements for the existence of limit cycle.

Theorem 4 (Liu [42]) *Consider any one of the parameters from the system 2, say h (predators' hunting cooperation) as the bifurcation parameter around the co-existing equilibrium, then the necessary and sufficient conditions for Hopf bifurcation are*

$$(a)\mu_1, \mu_2 > 0,\ \mu_3 = \mu_1\mu_2, \qquad (b)\frac{d}{dh}(\mu_1\mu_2 - \mu_3)_{h=h_H} \neq 0.$$

Proof If the point h_H is the Hopf bifurcation point, then the following condition must be satisfied:
(a)$\mu_1, \mu_2 > 0, \quad \mu_3 = \mu_1\mu_2$, (b)$\frac{d}{dh}(Re\lambda_i(h))_{h=h_H} \neq 0$ for $i = 1, 2$.
The characteristic equation as given in 4,

$$\lambda^3 + \mu_1\lambda^2 + \mu_2\lambda + \mu_3 = 0$$

i.e., $\lambda^3 + \mu_1\lambda^2 + \mu_2\lambda + \mu_1\mu_2 = 0$ [from the above assumption]

i.e., $(\lambda + \mu_1)(\lambda^2 + \mu_2) = 0$ (5)

i.e., $\lambda = -\mu_1, \ \pm\sqrt{\mu_2}\,i$

Thus, the eigenvalues would be—one negative, and one pair of purely imaginary if $\mu_1, \mu_2 > 0$.

The general form of roots of 4 for $h \in (h_H - \epsilon, h_H + \epsilon)$:

$$\lambda_1 = \chi_1 + \chi_2\, i, \quad \lambda_2 = \chi_1 - \chi_2\, i, \ ; \ \lambda_3 = -\mu_1.$$

Substituting $\lambda_1 = \chi_1 + \chi_2\, i$ in 5, we get,
$(\chi_1 + \chi_2\, \iota + \mu_1)(\chi_1^2 - \chi_2^2 + \iota\, \chi_1\chi_2 + \mu_2) = 0.$
Separating the real and imaginary part,
$\gamma_1(h)\chi_1' - \gamma_2(h)\chi_2' + \delta_1(h) = 0, \qquad \gamma_2(h)\chi_1' + \gamma_1(h)\chi_2' + \delta_2(h) = 0.$
where $\gamma_1 = 3\chi_1^2 - 3\chi_2^2 + \mu_2 + 2\mu_1\chi_1,$
$\gamma_2 = 6\chi_1\chi_2 + 2\chi_2\mu_1,$
$\delta_1 = \mu_1'(\chi_1^2 - \chi_2^2) + \mu_2'\chi_1 + \mu_3',$
$\delta_2 = \mu_2'\chi_2 + 2\mu_1'\chi_1\chi_2.$
But we have, $\chi_1(h_H) = 0, \ \chi_2(h_H) = \sqrt{\mu_2(h_H)}i,$
$\therefore \ \gamma_1(h_H) = -2\mu_2, \ \gamma_2(h_H) = 2\mu_1\sqrt{\mu_2}, \ \delta_1(h_h) = \mu_3' - \mu_1'\mu_2, \ \delta_2(h_H) = \sqrt{\mu_2}\mu_2'.$

$$\text{Hence, } \frac{d}{dh}[Re(\lambda_{1,2}(h))]_{h=h_H} = \chi_1 \quad = -\frac{\gamma_1\delta_1 + \gamma_2\delta_2}{\gamma_1^2 + \gamma_2^2}$$
$$= -\frac{\mu_1\mu_2' + \mu_1'\mu_2 - \mu_3'}{2\mu_2 + 2\mu_1^2}$$
$$\neq 0,$$
$$\text{if } \mu_1\mu_2' + \mu_1'\mu_2 - \mu_3' \neq 0.$$

Hence, our theorem.

6 Numerical Simulation

With the aid of MATLAB and MatCont 6 software [43], the disquisition of the rich dynamical scenarios of the system of Eq. 2 is the focus of this section. Through alteration of the parametric values of a_1 (the immature's rate of predation), a_2 (the matured's rate of predation), b (negative repercussion of immature's predation), h (matured predators' hunting cooperation), s (impact of fear due to hunting cooperation), their ascendancy over the system are visualized. For the parameters, hypothetical and biologically viable numerical values are taken from Table 1.

With the parametric values from Table 1, Fig. 1 is visualized depicting the local stability of the co-existing equilibrium point $(x^*, y^*, z^*) = (0.133, 0.18, 0.3)$ through the phase portrait. Theoretically, as given in Theorem 3, the Routh-Hurwitz criterion too is satisfied, since we have, $\mu_1 = 0.4327 > 0$, $\mu_3 = 0.0121 > 0$, $\mu_1\mu_2 - \mu_3 = 0.00418 > 0$. Thus, the corroboration of co-existing equilibrium (0.133, 0.18, 0.3) being locally asymptotically stable is done through both analytical and graphical methodology.

Authentication of existence of limit cycle as formulated in Theorem 4 is done by varying the parameter h (the matured predators' hunting cooperation). For $h = h_H = 0.833582$, we have $\mu_1 = 0.423524$, $\mu_3 = 0.0129336\,\mu_1\mu_2 - \mu_3 = 0$, and

Table 1 Parametric values

Physical meaning	Parameter	Value
Predation rate by immature predators	a_1	0.6
Predation rate by matured predators	a_2	1.1
Conversion rate of killed prey into immature predator	u	1.2
Transition rate of immature predators into matured stage	c	0.15
Growth rate of prey	r	0.5
Environment's carrying capacity	k	2.5
Natural death rate of immature predators	d_1	0.12
Natural death rate of matured predators	d_2	0.09
Death of immature predators due to their inefficiency	b	0.35
Hunting cooperation of matured predators	h	0.3
Impact of fear due to hunting cooperation	s	0.2

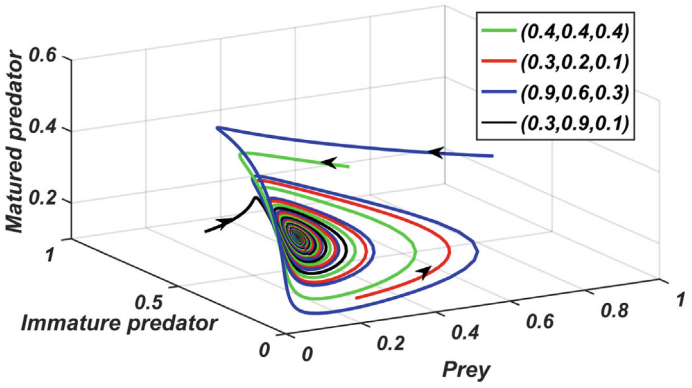


Fig. 1 Local stability of the co-existing equilibrium point

$\left[\frac{d}{dh}(\mu_1\mu_2 - \mu_3)\right]_{h_H} \neq 0$, therefore analytically Hopf bifurcation point for matured predators' hunting cooperation, h is h_H, which is visualized graphically in Fig. 2. Figure 2c delineates the co-existing equilibrium being stable for $h = 0.8 < h_H$, while Fig. 2b depicts the stable limit cycle being born at $h = h_H$, and Fig. 2a depicts a stable limit cycle surrounding the formerly stable co-existing equilibrium point.

The system of Eq. 2 also harbors Hopf bifurcation under variation of the predation rate of immature predators, a_1. Figure 3 illustrates the scenario of Hopf bifurcation, which is supercritical in nature owing to negative Lyapunov coefficient ($-1.157926e^{-01}$, which is obtained using the MatCont [43]), through phase portraits in the vicinity of the bifurcating point, i.e., $a_1 = a_{1H} = 0.380479$. From the Fig. 3a, it is clear that, for $a_1 > a_{1H}$ the system of equations possesses a stable co-existing equilibrium point, in view of the fact that the trajectory spirals into the equilibrium point. Figure 3b, c depict the trajectory spiraling around the now unstable co-existing equilibrium point for $a_1 = a_{1H}$ and $a_1 < a_{1H}$ respectively, also the increase in the circumference of the limit cycle is commensurate to the decrease of the parametric value of a_1.

To investigate the bifurcation scenarios corresponding to the predation rate of matured predators, a_2, Fig. 4 is visualized. As the parameter a_2 is varied, transcritical bifurcation, saddle node bifurcation, and Hopf bifurcation are encountered. The bifurcation where the stabilities of two different equilibrium points is swapped over, identified as transcritical bifurcation, is at $a_2 = a_{2t} = 0.229$, where for $a_2 < a_{2t}$ predator free equilibrium point is stable, contrariwise for $a_2 > a_{2t}$ co-existing equilibrium point is stable. The saddle node bifurcation is found at $a_2 = a_{2s} = 0.168539$, where

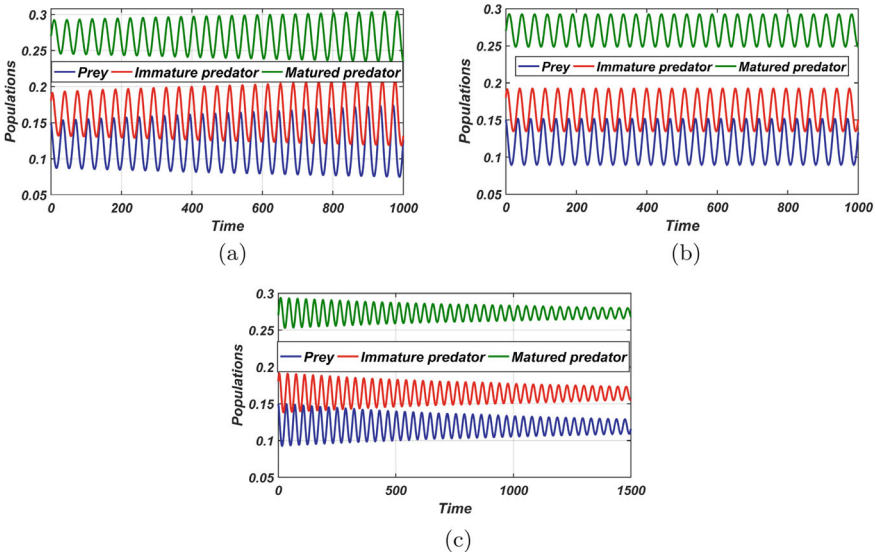


Fig. 2 Time series for different parametric values of matured predators' hunting cooperation, h: **a** $h = 0.85(> h_H)$, **b** $h = 0.832582(= h_H)$, **c** $h = 0.8(< h_H)$. h_H is the Hopf bifurcation point

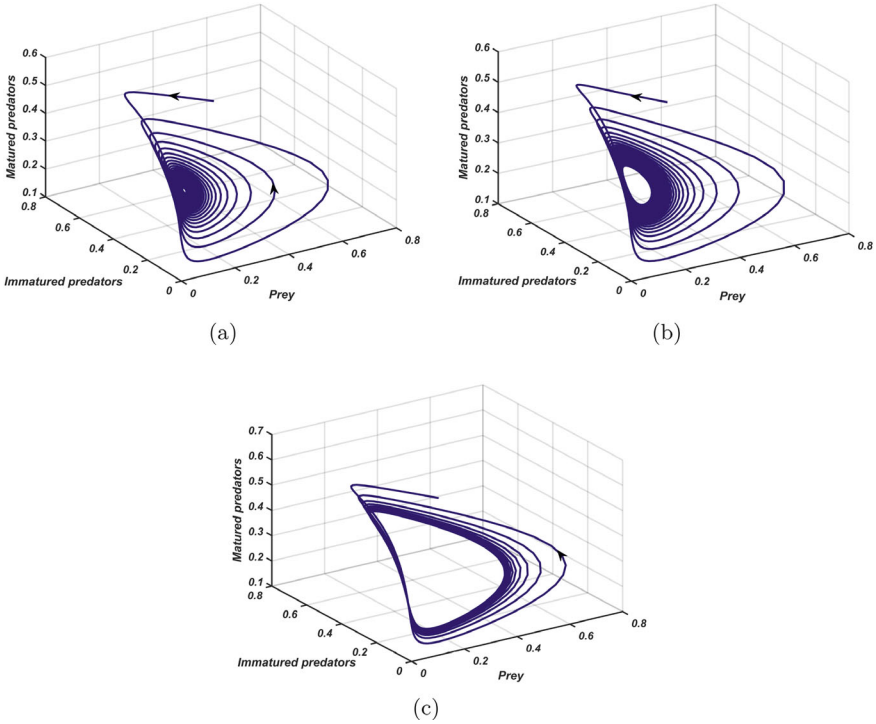


Fig. 3 Phase portraits for different parametric values of predation rate of immature predators, a_1: **a** $a_1 = 0.42(> a_{1H})$, **b** $a_1 = 0.380499(= a_{1H})$, **c** $a_1 = 0.3(< a_{1H})$. a_{1H} is the Hopf bifurcation point

a stable co-existing equilibrium merges with another co-existing equilibrium which is unstable. Also, because of existence Hopf bifurcation at $a_2 = a_{2H} = 1.542792$, which is supercritical owing to the Lyapunov coefficient $-1.497119e^{-01}$ (which is obtained using the MatCont [43]), i.e., a negative Lyapunov coefficient, the stability of co-existing equilibrium is interchanged with the limit cycle thus born. The amplitude of the limit cycle augments with an increase in the parametric value beyond the threshold value.

As can be described from Fig. 4, the bistability scenario can be evidenced for a_2 in the region (a_{2s}, a_{2t}), i.e., the scenario where both the equilibriums, predator free equilibrium and co-existing equilibrium are locally and asymptotically stable. The determination of which one of equilibrium point is to be attained through time series is strongly dependent on the initial populations of prey species and both of the stages of predator species. Similarly for the parameter b (death rate of immature predators due to their inefficiency), at $b = 2.092$ there is a reciprocity of stability between predator free equilibrium and co-existing equilibrium through transcritical bifurcation, for $b > 2.092$ the predator free equilibrium is stable, contrariwise the co-existing equilibrium is stable; at $b = 2.106268$ a saddle node bifurcation occurs,

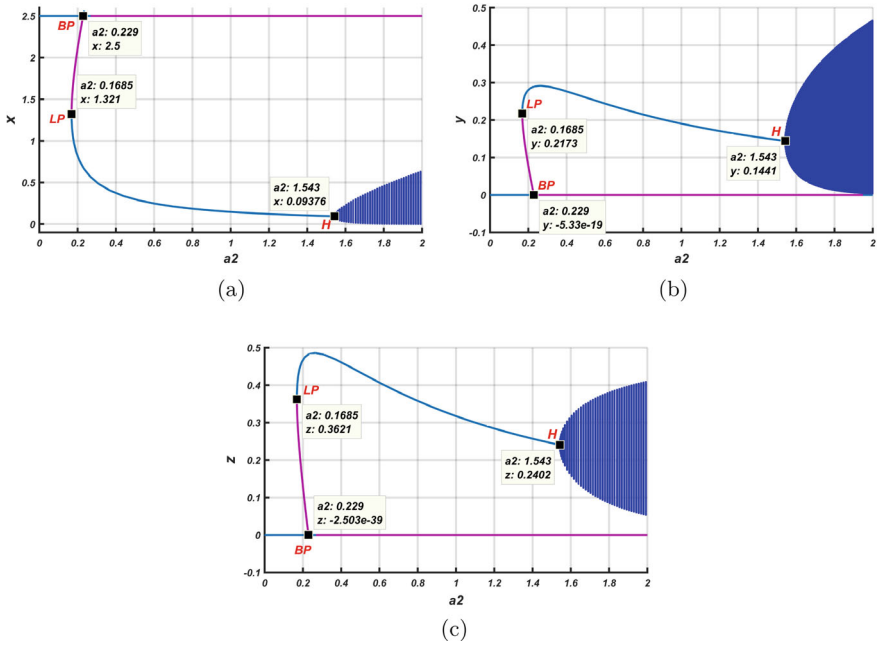
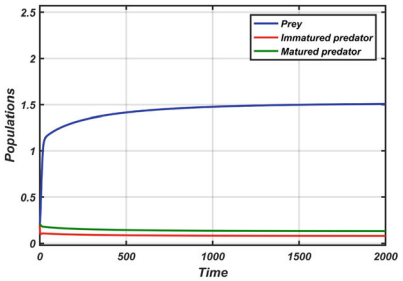


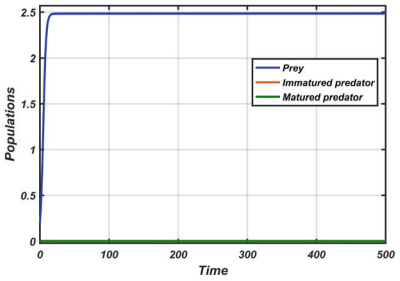
Fig. 4 Dependence of **a** prey population, **b** immature predator population, **c** matured predator population on predation rate of matured predator, a_2 regarding their equilibrium state. H denotes Hopf bifurcation point, saddle node bifurcation point is signified by LP, and the transcritical bifurcation point by BP. Solid blue line denotes stabile equilibriums, and solid purple line denotes the unstable equilibriums

where an unstable co-existing equilibrium and a stable co-existing equilibrium collide and annihilate each other, for $b < 2.106268$ there is a stable coexisting equilibrium point. Thereupon, for $b \in (2.092,\ 2.106268)$ we have a stable predator free equilibrium and a stable co-existing equilibrium, thus a bi-stable region. Figure 5 visualizes this very scenario by taking different initial populations at $b = 2.1$. For the initial population $(0.2, 0.2, 0.2)$, the co-existing population is espied to be stable (Fig. 5a), for initial populations $(0.2, 0.01, 0.001)$ and $(0.00002, 0.2, 0.2)$, the predator free equilibrium point is stable (Fig. 5b, c). Therefore, the predator free equilibrium is attainable only when the prey's initial population or the predators' initial population are considerably low.

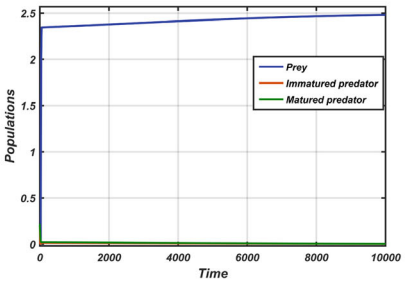
Figure 6 visualizes the equilibrium states with respect to all the populations, associated with the variation in the effect of fear owing to matured predators' hunting cooperation (s). The system of Eq. 2 experiences supercritical Hopf bifurcation at $s = 0.8949$ when all other parametric values are taken from Table 1 except $u = 2.2$. For $0 < s < 0.8949$, the system pursues a stable trajectory denoted by solid blue line, while for $s \geq 0.8949$ the system is described to oscillate, which is beheld to be escalating with the increase of parametric value.



(a) with initial population $(0.2, 0.2, 0.2)$.



(b) with initial population $(0.2, 0.01, 0.001)$.



(c) with initial population $(0.00002, 0.2, 0.2)$.

Fig. 5 Bi-stability between predator free equilibrium and co-existing equilibrium owing to different initial populations at $b = 2.1$

7 Conclusion

In the paper, the predator-prey interactive dynamic was explored. The predator species had two stages, which were immature stage and matured stage, and both the stages of the predator evinced the caliber for predation. Owing to the importance of characteristics and variability of hunting cooperation and fear effect in a predator-prey system, both of them were embodied in the model. Their amalgamated effect on the whole system was investigated. Also synchronically, the immature predators were racked with mortal peril as a negative repercussion of their predation, by virtue of lack of experience in predation process and also their physique when compared to that of the matured predators. For the same reason, cooperative hunting was assumed only in the matured predators. One such biological scenario analogous to our model is that of the interaction between the wolves and the elks.

Three types of equilibrium points were maintained by the system of Eq. 2: vanishing equilibrium $(0, 0, 0)$, predator free equilibrium $(k, 0, 0)$, and co-existing equilibrium (x^*, y^*, z^*). To discern the conduct of the system of Eq. 2 in the vicinity of all three equilibrium points, through Theorems 1, 2, 3, the local stabilities were investigated. Also, the requisites for Hopf bifurcation were acquired analytically

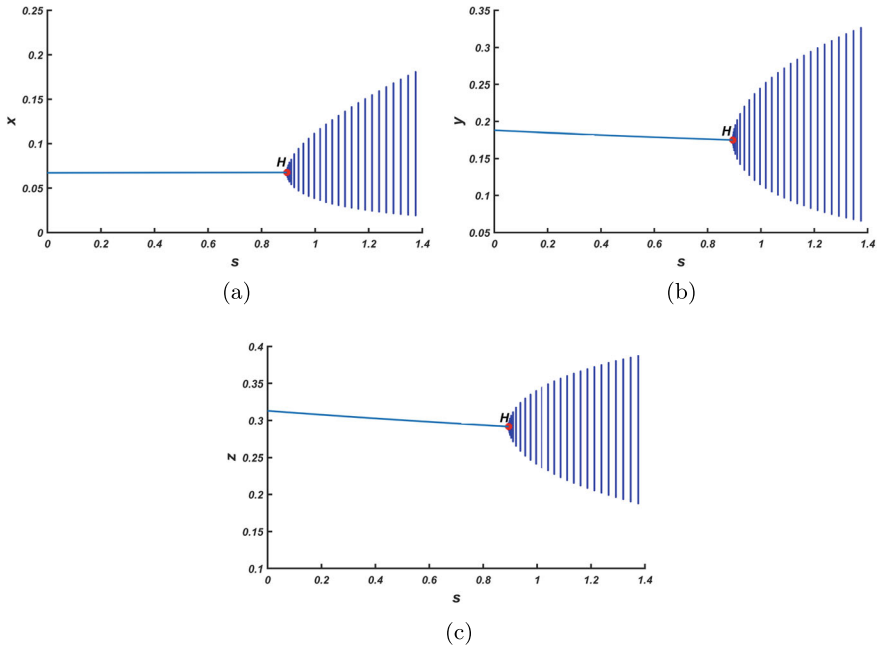


Fig. 6 Dependence of **a** prey population, **b** immature predator population, **c** matured predator population on impact of fear, s regarding their equilibrium state. The Hopf bifurcation point is at $s_H = 0.8949$

(Theorem 4), which, via numerical simulation, was consolidated to be supercritical, i.e., the stability of the co-existing equilibrium point was transported to the spawned limit cycle (Figs. 2 and 3). Other bifurcations such as transcritical bifurcation and saddle node bifurcation were also found (Fig. 4). Bistability was found between predator free equilibrium and co-existing equilibrium. During the bi-stability phenomenon, predator free equilibrium point can be achieved only when the initial populace of either prey species or predator species was considerably low (Fig. 5).

When the cooperative hunting was really strong, that is to say when the parameter for hunting cooperation (h) was augmented beyond a threshold value, none of the species was effectual in conforming to a specific population size, and thus the oscillation. The reason being, with a high cooperative hunting the population of predator species surges, while at the same time more prey were killed.

Similarly, an enormous impact of fear caused the prey biomass to remarkably decrease, which inevitably steered the predator species population to diminish because of their dependence on the prey species. Again, with low biomass of predator, the prey species amplified their total biomass, the consequence of which was the escalation of predator species all over again, thus the resulting undulation.

Equivalently, the prepotency of other vital parameters such as the predation rate of immature predator (a_1), the death rate of immature predator due to counter-attack of the prey (b), and the mature predator's predation rate (a_2) were deliberated upon.

Competing Interest The authors declare no competing interest.

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Bianchi Type I Cosmological Model with Time-Dependent Cosmological Constant in Brans-Dicke Theory of Gravity

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Abstract In this chapter, we explore cosmic dynamics in a spatially homogeneous and anisotropic Bianchi type I universe within the framework of Brans-Dicke theory of gravity with a time varying cosmological constant. We assume that the average scale factor a obeys a combination of a power law and exponential law, known as hybrid expansion law and obtain exact solution of the field equations by considering a power law relation between the Brans-Dicke scalar field ϕ and the scale factor a. We studied the evolution of relevant cosmological parameters and checked the validity of the four energy conditions. The statefinder diagnostic pair of the model is also obtained. We find that the model exhibits the cosmic dynamics which is in agreement with the current observational data.

1 Introduction

From various recent cosmological and astrophysical observations the current understanding of our universe reveals that the universe, which is nearly 13.8 billion years old, is currently passing through a phase of accelerated expansion [1–7] the root cause of which is yet unknown. Within the framework of General Relativity, the observed late time cosmic acceleration should be due to the presence of a gravitationally repulsive physical entity with large negative pressure that occupies nearly 68.3% of the total content of the universe. But the true nature of this hypothetical physical entity, dubbed dark energy, causing the universe to expand with an increasing rate is still a great mystery. Although the General Theory of Relativity has been able to explain a number of cosmological phenomena including the evolution of the universe to a very striking accuracy, it fails to explain the observed phenomena of the accelerated

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expansion of the universe in a convincing way. So, much interest has been grown in the literature to investigate this challenging problem of cosmology with modified theories of gravity. Some of the popular modified theories of gravity are the scalar-tensor theories, especially the Brans-Dicke theory of gravity [8], the $f(R)$ gravity [9], $f(R,T)$ gravity [10], f(T) gravity [11], $f(G)$ gravity [12], $f(R,G)$ gravity [13], $f(Q)$ gravity [14], $f(Q,T)$ gravity [15], etc.

The Brans-Dicke theory of gravity is one of the most natural modifications of the General Theory of Relativity. Brans and Dicke developed this theory by absorbing Mach's Principle in the theory of relativity [16]. In this theory, gravity couples with a time-dependent scalar field ϕ through a constant coupling parameter ω. The scalar field ϕ plays the role of inverse of the gravitational constant G and the General Theory of Relativity can be recovered from the Brans-Dicke theory when $\omega \to \infty$ and ϕ is constant. ϕ also plays an important role in describing early inflationary era and the late-time accelerated expansion phase of the universe. The Brans-Dicke theory of gravity has passed the experimental tests from solar system [17], CMB (Cosmic Microwave Background) data [18] and Planck data [19]. This theory is therefore widely studied by many authors in different contexts. Several authors have also studied the dynamics of both flat and non-flat universe described by the FLRW (Friedmann-Lemaître-Robertson-Walker) metric with time-dependent cosmological constant $\Lambda(t)$ in Brans-Dicke theory of gravity. But the small anisotropies observed in some cosmological observations such as Wilkinson Microwave Anisotropy Probe (WMAP) [4], Large Scale Structure (LSS) [20] and Cosmic Microwave Background (CMB) radiation [21] indicate that although our universe appears to be spatially homogeneous and isotropic at large scales, the FLRW cosmological model failed to explain the early phases of the cosmic evolution properly. So naturally, spatially homogeneous and anisotropic models of the universe are to be studied to explore some such features of the universe. Bianchi type models are the simplest homogeneous and anisotropic models of the universe which are studied in the literature in the context of dark energy and also in modified theories of gravity. Recently, Yadav et al. [22] have investigated a Bianchi type-I transitioning universe with a hybrid scalar field in Brans-Dicke theory of gravity and found that the derived model of the universe validates Mach's principle and is consistent with the recent observational data.

In literature a number of cosmological models have been proposed by the researchers to study the cosmic expansion. Mishra and Dua examined the flat FLRW universe with time varying cosmological constant and Bianchi type-I model in Brans-Dicke theory of gravity [23, 24]. Singh et al. [25] studied the interacting and also non-interacting dark energy and dark matter in the scope of spatially homogeneous and anisotropic Bianchi type-I universe in Brans-Dicke theory, Naidu et al. [26] discussed the Bianchi type-V dark energy cosmological model in presence of an attractive massive scalar field in general relativity, Mishra and Dua [27] considered Sáez-Ballester theory of gravity to investigate Bianchi type-I cosmological model with variable deceleration parameter, Venkateswarlu and Satish [28] investigated LRS Bianchi type-I model coupled with scalar field. Hegazy [29] have obtained exact solutions of Bianchi type-I cosmological model in the self-creation theory of gravitation. Rao and Neelima [30] have presented a LRS Bianchi type-I dark energy

cosmological models in the Nordtvedt general scalar tensor theory of gravitation. Mishra and Dua [31] studied the $f(R, T)$ gravity theory in a LRS Bianchi type-I bulk viscous universe and have done a comparison between the two models with deceleration parameters, one varying linearly another varying bilinearly. Tripathy et al. [32] discussed a bouncing scenario in the Brans-Dicke theory using an exponential scale factor. Bharadwaj and Rana [33] studied Lyra's cosmology in the Bianchi type-III universe using a generalized hybrid expansion law. Furthermore, several researcher ([34–38]) considered a hybrid scale factor in exploring the evolving universe in various contexts.

In this work, we consider a spatially homogeneous and anisotropic Bianchi type-I universe within the framework of Brans-Dicke theory of gravity with a time-dependent cosmological constant and explore its cosmic dynamics. We organize the paper as follows: In Sect. 2, we derive the Brans-Dicke field equations corresponding to the Bianchi type-I space-time. In Sect. 3, we obtain cosmological solutions of Brans-Dicke field equations by assuming that the scale factor a obeys a hybrid expansion law, and by using a power law relation between the Brans-Dicke scalar field ϕ and the scale factor a. We study cosmic dynamics of the model by graphically representing some important cosmological parameters in Sect. 4. The statefinder diagnostic pair is also obtained and examined the validity of the energy conditions. In Sect. 5, the physical behaviours of the cosmological parameters are discussed in terms of redshift z. We conclude the paper in Sect. 6 by briefly discussing the obtained results.

2 Metric and the Field Equations

We take the spatially homogeneous and anisotropic Bianchi type I line-element in the form

$$ds^2 = dt^2 - A^2dx^2 - B^2dy^2 - C^2dz^2 \tag{1}$$

where A, B, C are the directional scale factors and are functions of the cosmic time t only.

We consider the Brans-Dicke action in the form

$$S = \int [\phi(R - 2\Lambda) + 16\pi\mathscr{L} + \frac{\omega}{\phi}\phi_{,i}\phi^{,i}]\sqrt{-g}d^4x \tag{2}$$

where R is the Ricci scalar, Λ is a time-dependent cosmological constant, $\mathscr{L}$ is the matter-Lagrangian density, ω is the dimensionless Brans-Dicke coupling constant, ϕ is a scalar field.

By taking the variations of the action (2) with respect to the metric tensor components g_{ij} and scalar field ϕ, we obtain the Brans-Dicke field equations as

$$R_{ij} - \frac{R}{2}g_{ij} + \Lambda g_{ij} = -\frac{8\pi T_{ij}}{\phi} - \frac{\omega}{\phi^2}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) - \frac{1}{\phi}(\phi_{,i;j} - g_{ij}\Box\phi) \tag{3}$$

$$\Box\phi = \frac{8\pi T + 2\Lambda\phi}{(3+2\omega)} \tag{4}$$

where $\phi_{,i}$ is the covariant derivative of the scalar field ϕ, R_{ij} is the Ricci tensor, g_{ij} is the metric tensor, T_{ij} is the energy-momentum tensor, $\Box = \nabla_k\nabla^k$ is the D'Alembert operator, $T = g^{ij}T_{ij}$ is the trace of the energy-momentum tensor and semicolon (;) represents the covariant derivative.

The energy-momentum tensor for a perfect fluid is given by

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij} \tag{5}$$

where ρ is the energy density and p is the pressure of the perfect fluid filling the universe.

For the line element (1), the Brans-Dicke field equations (3) and (4) are obtained as

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\frac{\ddot{\phi}}{\phi} - \Lambda = -\frac{8\pi p}{\phi} \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\frac{\ddot{\phi}}{\phi} - \Lambda = -\frac{8\pi p}{\phi} \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\ddot{\phi}}{\phi} - \Lambda = -\frac{8\pi p}{\phi} \tag{8}$$

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\dot{C}}{C}\frac{\dot{A}}{A} - \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \Lambda = -\frac{8\pi\rho}{\phi} \tag{9}$$

$$(3+2\omega)\left(\frac{\ddot{\phi}}{\phi} + 3\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\frac{\dot{\phi}}{\phi}\right) - 2\Lambda = -\frac{8\pi\ (\rho - 3p)}{\phi} \tag{10}$$

From Eqs. (6), (7) and (8), we have

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)\left(\frac{\dot{C}}{C} + \frac{\dot{\phi}}{\phi}\right) = 0 \tag{11}$$

$$\frac{\ddot{C}}{C} - \frac{\ddot{A}}{A} + \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right)\left(\frac{\dot{B}}{B} + \frac{\dot{\phi}}{\phi}\right) = 0 \tag{12}$$

$$\frac{\ddot{C}}{C} - \frac{\ddot{B}}{B} + \left(\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right)\left(\frac{\dot{A}}{A} + \frac{\dot{\phi}}{\phi}\right) = 0 \tag{13}$$

Adding Eqs. (11) and (12) we obtain

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}-2\frac{\ddot{A}}{A}+2\frac{\dot{B}}{B}\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-2\frac{\dot{A}}{A}\right)\frac{\dot{\phi}}{\phi}=0 \tag{14}$$

On integration Eqs. (11) and (14) yield

$$\frac{B}{A}=e^{\int\frac{kdt}{\phi a^3}}=D(t), \text{ say} \tag{15}$$

and

$$\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-2\frac{\dot{A}}{A}=\frac{h}{\phi ABC} \tag{16}$$

where k and h are constants of integration (Figs. 5, 7, 8, 12, 13, 15, 21, 23, 24, 28, 29 and 31).

Now, for a singular universe having a point type Big Bang singularity at time $t=0$, all the three directional scale factors vanish and for a non-singular universe, the directional scale factors become constant at $t=0$. Thus, applying these initial conditions, we can choose $h=0$. Hence, from Eq. (16), we obtain

$$\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-2\frac{\dot{A}}{A}=0 \tag{17}$$

Integration yields

$$BC=A^2 \tag{18}$$

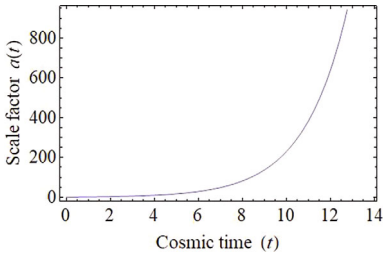

Fig. 1 Evolution of the scale factor a v/s cosmic time t for $\alpha=0.229$, $\beta=0.491$, $a_0=1$

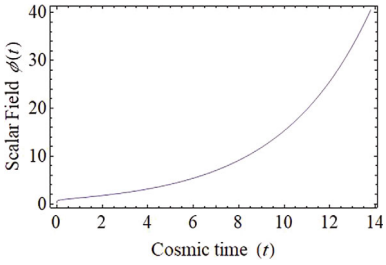

Fig. 2 Evolution of the scalar field ϕ v/s cosmic time t for $\alpha=0.229$, $\beta=0.491$, $m=0.502$, $a_0=\phi_0=1$

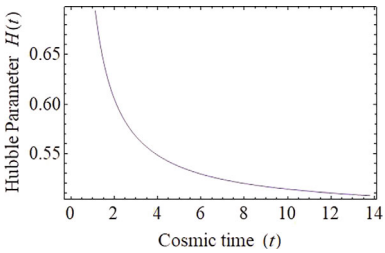

Fig. 3 Evolution of the Hubble parameter H v/s cosmic time t for $\alpha = 0.229, \beta = 0.491$

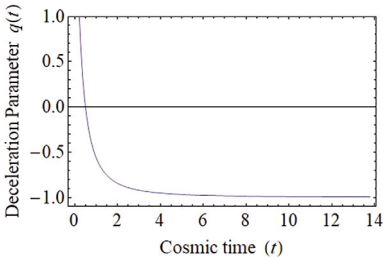

Fig. 4 Evolution of the deceleration parameter q v/s cosmic time t for $\alpha = 0.229, \beta = 0.491$

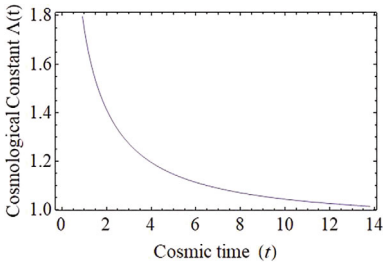

Fig. 5 Evolution of the cosmological constant Λ v/s cosmic time t for $\alpha = 0.229, \beta = 0.491, m = 0.502, \omega = 1.307, a_0 = \phi_0 = k = 1$

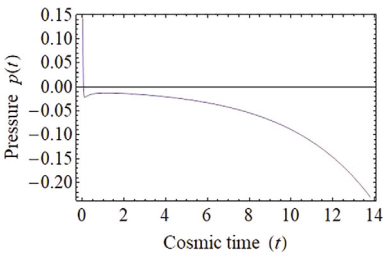

Fig. 6 Evolution of the pressure p v/s cosmic time t for $\alpha = 0.229, \beta = 0.491, m = 0.502, \omega = 1.307, a_0 = \phi_0 = 1$

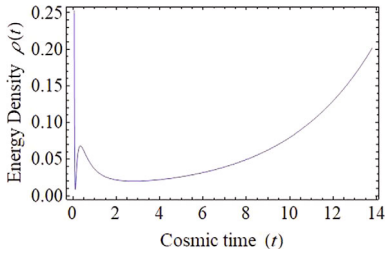

Fig. 7 Evolution of the energy density ρ v/s cosmic time t for $\alpha = 0.229$, $\beta = 0.491$, $m = 0.502$, $\omega = 1.307$, $a_0 = \phi_0 = k = 1$

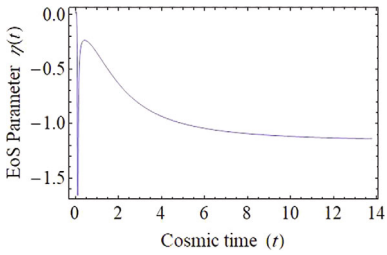

Fig. 8 Evolution of the EoS parameter η v/s cosmic time t for $\alpha = 0.229$, $\beta = 0.491$, $m = 0.502$, $\omega = 1.307$, $a_0 = \phi_0 = k = 1$

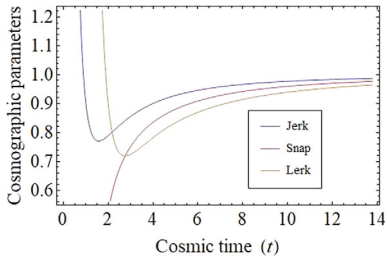

Fig. 9 Evolution of the cosmographic parameters v/s cosmic time t for $\alpha = 0.229$, $\beta = 0.491$

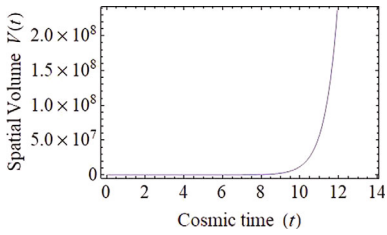

Fig. 10 Evolution of the spatial volume V v/s cosmic time t for $\alpha = 0.229$, $\beta = 0.491$, $a_0 = 1$

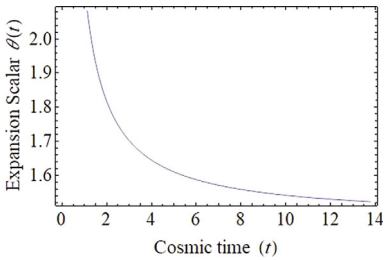

Fig. 11 Evolution of the expansion scalar θ v/s cosmic time t for $\alpha = 0.229, \beta = 0.491$

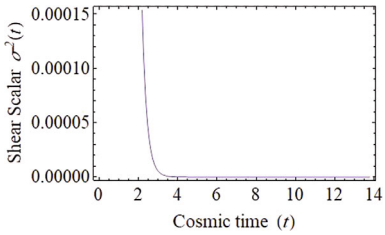

Fig. 12 Evolution of the shear scalar σ^2 v/s cosmic time t for $\alpha = 0.229, \beta = 0.491, m = 0.502, a_0 = \phi_0 = k = 1$

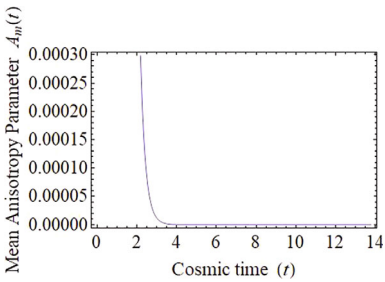

Fig. 13 Evolution of the mean anisotropy parameter A_m v/s cosmic time t for $\alpha = 0.229, \beta = 0.491, m = 0.502, a_0 = \phi_0 = k = 1$

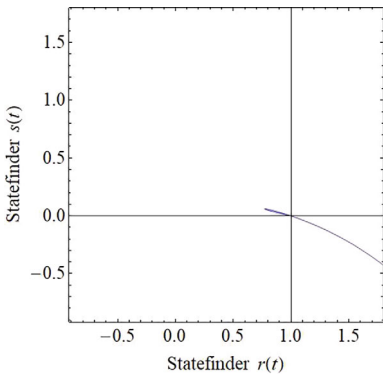

Fig. 14 Plot of the statefinder pair $\{r(t), s(t)\}$ for $\alpha = 0.229, \beta = 0.491$

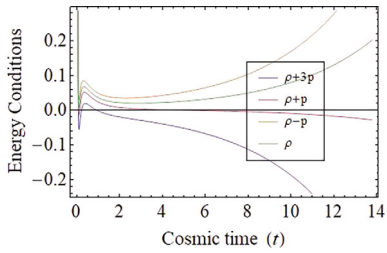

Fig. 15 Plot of the energy conditions v/s cosmic time t for $\alpha = 0.229$, $\beta = 0.491$, $m = 0.502$, $\omega = 1.307$, $a_0 = \phi_0 = k = 1$

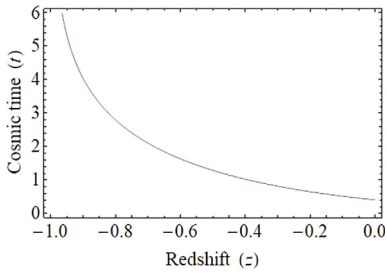

Fig. 16 Plot of the cosmic time t v/s redshift z for $\alpha = 0.229$, $\beta = 0.491$

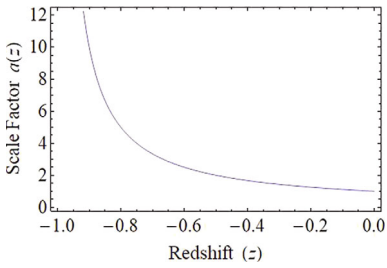

Fig. 17 Evolution of the scale factor a v/s redshift z

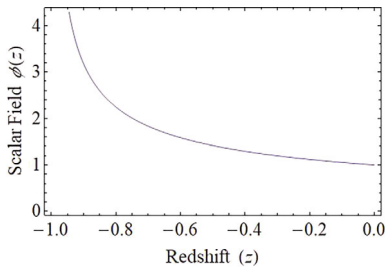

Fig. 18 Evolution of the scalar field ϕ v/s redshift z for $m = 0.502$, $\phi_0 = 1$

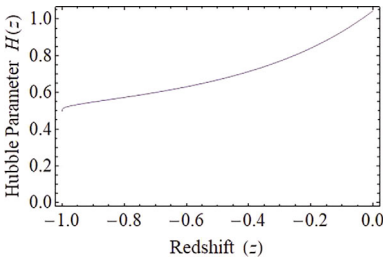

Fig. 19 Evolution of the Hubble parameter H v/s redshift z for $\alpha = 0.229, \beta = 0.491$

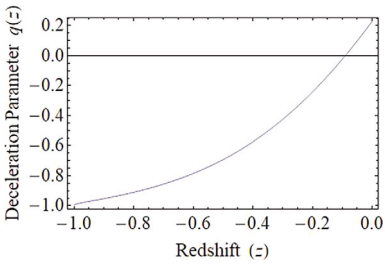

Fig. 20 Evolution of the deceleration parameter q v/s redshift z for $\alpha = 0.229, \beta = 0.491$

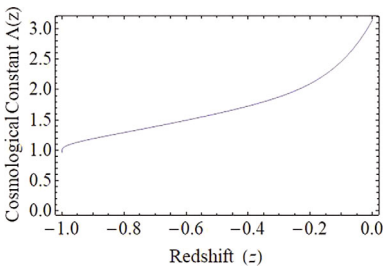

Fig. 21 Evolution of the cosmological constant Λ v/s redshift z for $\alpha = 0.229, \beta = 0.491, m = 0.502, \omega = 1.307, \phi_0 = k = 1$

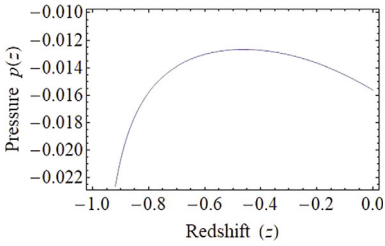

Fig. 22 Evolution of the pressure p v/s redshift z for $\alpha = 0.229, \beta = 0.491, m = 0.502, \omega = 1.307, \phi_0 = 1$

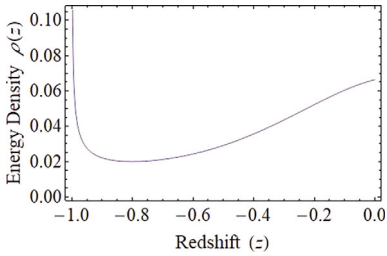

Fig. 23 Evolution of the energy density ρ v/s redshift z for $\alpha = 0.229$, $\beta = 0.491$, $m = 0.502$, $\omega = 1.307$, $\phi_0 = k = 1$

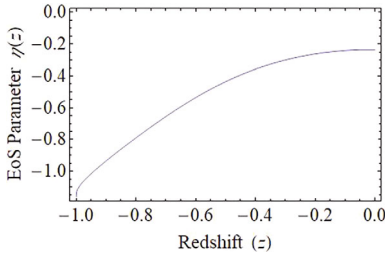

Fig. 24 Evolution of the EoS parameter η v/s redshift z for $\alpha = 0.229$, $\beta = 0.491$, $m = 0.502$, $\omega = 1.307$, $\phi_0 = k = 1$

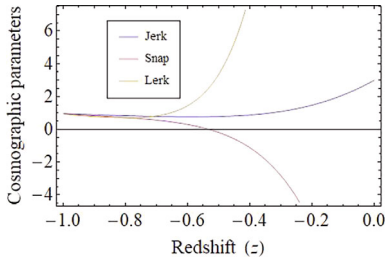

Fig. 25 Evolution of the cosmographic parameters v/s redshift z for $\alpha = 0.229$, $\beta = 0.491$

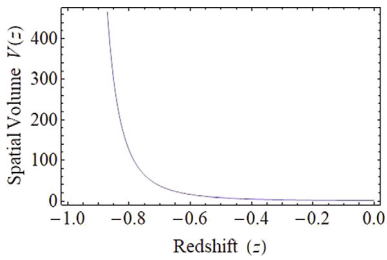

Fig. 26 Evolution of the spatial volume V v/s redshift z

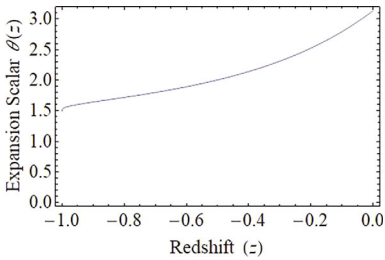

Fig. 27 Evolution of the expansion scalar θ v/s redshift z for $\alpha = 0.229, \beta = 0.491$

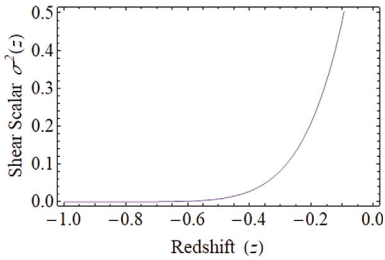

Fig. 28 Evolution of the shear scalar σ^2 v/s redshift z for $m = 0.502, \phi_0 = k = 1$

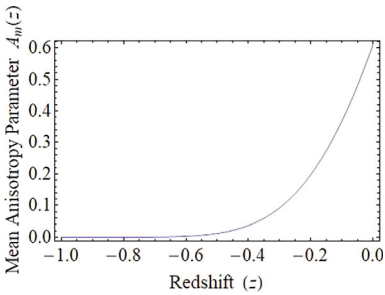

Fig. 29 Evolution of the mean anisotropy parameter A_m v/s redshift z for $\alpha = 0.229, \beta = 0.491, m = 0.502, \phi_0 = k = 1$

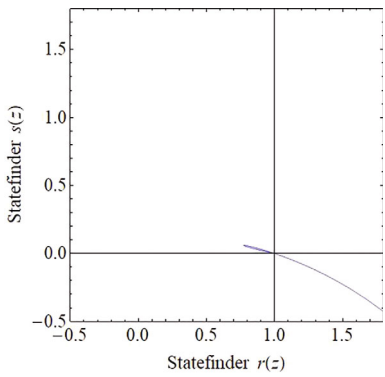

Fig. 30 Plot of the statefinder pair $\{r(z), s(z)\}$ for $\alpha = 0.229, \beta = 0.491$

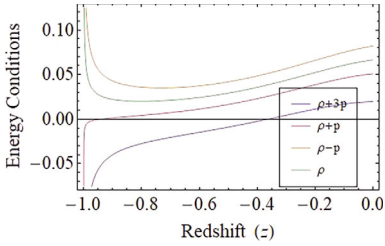

Fig. 31 Plot of the energy conditions v/s redshift z for $\alpha = 0.229$, $\beta = 0.491$, $m = 0.502$, $\omega = 1.307$, $\phi_0 = k = 1$

The average scale factor a for the metric (1) is defined as $a = (ABC)^{\frac{1}{3}}$. Therefore from (18), we obtain

$$a = A \tag{19}$$

So the directional scale factors B and C are obtained as $B = aD$ and $C = \frac{a}{D}$, where D is the measure of the anisotropy.
The field Eqs. (6)–(10) in terms of the average scale factor a can be rewritten as

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k^2}{a^6\phi^2} + 2\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\ddot{\phi}}{\phi} - \Lambda = -\frac{8\pi p}{\phi} \tag{20}$$

$$3\frac{\dot{a}^2}{a^2} - \frac{k^2}{a^6\phi^2} + 3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - \Lambda = \frac{8\pi\rho}{\phi} \tag{21}$$

$$(3+2\omega)\left(\frac{\ddot{\phi}}{\phi} + 3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}\right) - 2\Lambda = \frac{8\pi(\rho - 3p)}{\phi} \tag{22}$$

Using Eqs. (20), (21) and (22), the cosmological constant $\Lambda(t)$, pressure $p(t)$ and energy density $\rho(t)$ can be expressed as

$$\Lambda(t) = 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} + \frac{k^2}{a^6\phi^2} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - 3\omega\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \omega\frac{\ddot{\phi}}{\phi} \tag{23}$$

$$p(t) = \frac{\phi}{8\pi}\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - (2+3\omega)\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - (1+\omega)\frac{\ddot{\phi}}{\phi}\right] \tag{24}$$

$$\rho(t) = \frac{\phi}{8\pi}\left[-3\frac{\ddot{a}}{a} - 2\frac{k^2}{a^6\phi^2} - \omega\frac{\dot{\phi}^2}{\phi^2} + 3(1+\omega)\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + \omega\frac{\ddot{\phi}}{\phi}\right] \tag{25}$$

3 Cosmological Solutions of the Field Equations

We have three independent equations with five unknowns a, ϕ, Λ, ρ and p. In order to obtain an exact solution of the field equations, we assume the scale factor a to obey the hybrid expansion law

$$a = a_0 t^{\alpha} e^{\beta t} \tag{26}$$

where α and β are the non-negative constants and a_0 represents the present value of a and consider the scalar field ϕ to be a power law function of a given by

$$\phi = \phi_0 a^m \tag{27}$$

where ϕ_0 is a constant of proportionality and m is an arbitrary constant.
The relation (26) gives the power law expansion for $\beta = 0$ and exponential expansion for $\alpha = 0$, hence the name hybrid expansion law first proposed by Akarsu et al. [39]. Using (26) in (27), the scalar field ϕ can be expressed as

$$\phi = \phi_0 \left(a_0 t^\alpha e^{\beta t}\right)^m \tag{28}$$

Using relations (26) and (28) in (23), (24) and (25), we then obtain

$$\Lambda(t) = \left(\frac{\alpha}{t} + \beta\right)^2 \left(6 - \frac{\omega}{2} m^2 - 3\, m\omega\right) + \frac{\alpha}{t^2}\left(m\omega - 3\right) + \frac{k^2}{\phi_0^2 \left(a_0 t^\alpha e^{\beta t}\right)^{(2\,m+6)}} \tag{29}$$

$$p(t) = \frac{\phi_0}{8\pi}\left(a_0 t^\alpha e^{\beta t}\right)^m \left[\left(\frac{\alpha}{t} + \beta\right)^2 \left(3 - 2\,m - 3\,m\omega - m^2 - m^2\omega\right)\right] + \frac{\phi_0}{8\pi}\left(a_0 t^\alpha e^{\beta t}\right)^m \left[\frac{\alpha}{t^2}\left(m + m\omega - 1\right)\right] \tag{30}$$

$$\rho(t) = \frac{\phi_0}{8\pi}\left(a_0 t^\alpha e^{\beta t}\right)^m \left[3\left(\frac{\alpha}{t} + \beta\right)^2 \left(m + m\omega - 1\right) + \frac{\alpha}{t^2}(3 - m\omega)\right] - \frac{\phi_0}{8\pi}\left(a_0 t^\alpha e^{\beta t}\right)^m \left[2\frac{k^2}{\phi_0^2 \left(a_0 t^\alpha e^{\beta t}\right)^{(2\,m+6)}}\right] \tag{31}$$

4 Physical Behaviour of the Model

Our universe started with a Big Bang some 13.8 billion years ago and ever since it has been expanding. The Hubble parameter H determines the rate of this cosmic expansion. For the universe characterized by the line element (1), the average Hubble parameter H is defined by

$$H = \frac{1}{3}(H_1 + H_2 + H_3)$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$, $H_3 = \frac{\dot{C}}{C}$ are the directional Hubble parameters along the three spatial direction x, y, z respectively.

For our model, the average Hubble parameter H is obtained as

$$H(t) = \frac{\alpha}{t} + \beta \tag{32}$$

The deceleration parameter q is one more very important cosmological parameter which indicates whether the expansion of the universe is accelerating, uniform or decelerating. It is defined by

$$q = -1 - \frac{\dot{H}}{H^2}$$

and for our model q is obtained as

$$q = -1 + \frac{\alpha}{(\alpha + \beta t)^2} \tag{33}$$

The equation of state (EoS) parameter $\eta(t) = \frac{p(t)}{\rho(t)}$ is obtained as

$$\eta(t) = \frac{(\alpha+\beta t)^2(3-2m-3m\omega-m^2-m^2\omega)+\alpha(m+m\omega-1)}{3(\alpha+\beta t)^2(m+m\omega-1)+\alpha(3-m\omega)-2\frac{k^2t^2}{\phi_0^2(a_0t^\alpha e^{\beta t})^{(2m+6)}}} \tag{34}$$

The cosmographic parameters viz., the jerk (j), snap (s) and lerk parameter (l) [40], defined by

$$j(t) = \frac{1}{aH^3}\frac{d^3a}{dt^3}, s(t) = \frac{1}{aH^4}\frac{d^4a}{dt^4} \text{ and } l(t) = \frac{1}{aH^5}\frac{d^5a}{dt^5}$$

are obtained as

$$j(t) = \frac{2\alpha}{(\alpha+\beta t)^3} - \frac{3\alpha}{(\alpha+\beta t)^2} + 1 \tag{35}$$

$$s(t) = \frac{3\alpha(\alpha-2)}{(\alpha+\beta t)^4} + \frac{8\alpha}{(\alpha+\beta t)^3} - \frac{6\alpha}{(\alpha+\beta t)^2} + 1 \tag{36}$$

$$l(t) = \frac{4\alpha(6-5\alpha)}{(\alpha+\beta t)^5} + \frac{15\alpha(\alpha-2)}{(\alpha+\beta t)^4} + \frac{20\alpha}{(\alpha+\beta t)^3} - \frac{10\alpha}{(\alpha+\beta t)^2} + 1 \tag{37}$$

The spatial volume parameter $V = a^3$, the expansion scalar $\theta = 3H$, the shear scalar $\sigma^2 = 3\left(\frac{\dot{a}}{a}\right)^2 - \left(\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\dot{C}}{C}\frac{\dot{A}}{A}\right)$ and mean anisotropy parameter $A_m = \frac{1}{3H^2}\sum_{i=1}^{3}(H_i - H)^2$ are obtained in terms of the cosmic time t as

$$V(t) = \left(a_0 t^\alpha e^{\beta t}\right)^3 \tag{38}$$

$$\theta(t) = 3\left(\frac{\alpha}{t} + \beta\right) \tag{39}$$

$$\sigma^2(t) = \frac{k^2}{\phi_0^2\left(a_0 t^\alpha e^{\beta t}\right)^{(2m+6)}} \tag{40}$$

$$A_m(t) = \frac{2}{3}\left(\frac{\alpha}{t} + \beta\right)^{-2} \frac{k^2}{\phi_0^2\left(a_0 t^\alpha e^{\beta t}\right)^{(2m+6)}} \tag{41}$$

4.1 Graphical Representation of the Parameters in Terms of the Cosmic Time t

4.2 Statefinder Diagnostic

The dimensionless parameters r and s, introduced by Sahni et al. [41] are defined by

$$r = a^2 \frac{d^3a}{dt^3}\left(\frac{da}{dt}\right)^{-3} \text{ and } s = \frac{2}{3}\frac{(1-r)}{(1-2q)}$$

For our model, these values are obtained as

$$r(t) = \frac{2\alpha}{(\alpha+\beta t)^3} - \frac{3\alpha}{(\alpha+\beta t)^2} + 1 \tag{42}$$

$$s(t) = \frac{6\alpha(\alpha+\beta t) - 4\alpha}{9(\alpha+\beta t)^3 - 6\alpha(\alpha+\beta t)} \tag{43}$$

This dimensionless pair helps in analyzing different dark energy models. The dark energy models and ΛCDM models can be well distinguished using the pair.

The present value of the statefinder pair is $\{r = 1, s = 0\}$ which supports that our model behaves as ΛCDM one.

4.3 Energy Conditions

The four energy conditions are defined as
Strong Energy Condition (SEC) is $\rho + 3p \geq 0$, $\rho + p \geq 0$
Weak Energy Condition (WEC) is $\rho + p \geq 0$, $\rho \geq 0$
Dominant Energy Condition (DEC) is $\rho + p \geq 0$, $\rho - p \geq 0$, $\rho \geq 0$
Null Energy Condition (NEC) is $\rho + p \geq 0$

Among these four energy conditions, validation of the SEC hints the decelerated expansion of the universe and the violation hints the accelerated expansion of the

universe. WEC conveys that the density remains positive and is non-increasing in nature. DEC comes up with an upper bound for the energy density and as a result it becomes an upper bound for the expansion rate of the universe too. NEC is a guaranteed indication of the decrease in the density with the expanding universe.

5 Cosmolological Parameters in Terms of Redshift z

In this section, we discuss the time-dependent cosmological parameters of our model in terms of the redshift z.

The relation between the scale factor and the redshift is given by

$$a = \frac{a_0}{1+z} \tag{44}$$

where a_0 is the present value of a. Comparing the relations (26) and (44) with the present value of $a_0 = 1$, we obtain the cosmic time t and redshift z relation as

$$t = \frac{\alpha}{\beta} W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right] \tag{45}$$

where W is the Lambert W function.

Using the relations (44) and (45), we obtain the above obtained time-dependent parameters (Eqs. (27)–(43)) in terms of redshift z as

Scalar field,

$$\phi(z) = \phi_0\left(\frac{1}{1+z}\right)^m \tag{46}$$

Cosmological constant,

$$\Lambda(z) = \left(\frac{\beta}{W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]} + \beta\right)^2 \left(6 - \frac{\omega}{2}m^2 - 3m\omega\right) + \frac{\beta^2}{\alpha\left(W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^2}(m\omega - 3) + \frac{k^2}{\phi_0^2\left(\frac{1}{1+z}\right)^{(2m+6)}} \tag{47}$$

Pressure,

$$p(z) = \frac{\phi_0}{8\pi}\left(\frac{1}{1+z}\right)^m \left[\left(\frac{\beta}{W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]} + \beta\right)^2 (3 - 2m - 3m\omega - m^2 - m^2\omega)\right] + \frac{\phi_0}{8\pi}\left(\frac{1}{1+z}\right)^m \left[\frac{\beta^2}{\alpha\left(W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^2}(m + m\omega - 1)\right] \tag{48}$$

Energy density,

$$\rho(z) = \frac{\phi_0}{8\pi}\left(\frac{1}{1+z}\right)^m \left[3\left(\frac{\beta}{W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]} + \beta\right)^2 (m + m\omega - 1)\right] + \frac{\phi_0}{8\pi}\left(\frac{1}{1+z}\right)^m \left[\frac{\beta^2}{\alpha\left(W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^2}(3 - m\omega) - \frac{2k^2}{\phi_0^2\left(\frac{1}{1+z}\right)^{(2m+6)}}\right] \tag{49}$$

Hubble parameter,

$$H(z) = \frac{\beta}{W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]} + \beta \tag{50}$$

Deceleration parameter,

$$q(z) = -1 + \frac{1}{\alpha\left(1 + W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^2} \tag{51}$$

EoS parameter,

$$\eta(z) = \frac{p(z)}{\rho(z)} \tag{52}$$

can be obtained using (48) and (49), which is a lengthy expression.

Cosmographic parameters are obtained as

$$j(z) = \frac{2}{\alpha^2\left(1 + W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^3} - \frac{3}{\alpha\left(1 + W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^2} + 1 \tag{53}$$

$$s(z) = \frac{3(\alpha-2)}{\alpha^3\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^4} + \frac{8}{\alpha^2\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^3} - \frac{6}{\alpha\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^2} + 1 \tag{54}$$

$$l(z) = \frac{4(6-5\alpha)}{\alpha^4\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^5} + \frac{15(\alpha-2)}{\alpha^3\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^4} + \frac{20}{\alpha^2\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^3} - \frac{10}{\alpha\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^2} + 1 \tag{55}$$

Spatial volume,

$$V(z) = \left(\frac{1}{1+z}\right)^3 \tag{56}$$

Expansion scalar,

$$\theta(z) = 3\left(\frac{\beta}{W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]} + \beta\right) \tag{57}$$

Shear scalar,

$$\sigma^2(z) = \frac{k^2}{\phi_0^2\left(\frac{1}{1+z}\right)^{(2m+6)}} \tag{58}$$

Mean anisotropy parameter,

$$A_m(z) = \frac{2}{3}\left(\frac{\beta}{W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]} + \beta\right)^{-2}\left(\frac{k^2}{\phi_0^2\left(\frac{1}{1+z}\right)^{(2m+6)}}\right) \tag{59}$$

The statefinder pair in terms of z is obtained as

$$r(z) = \frac{2}{\alpha^2\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^3} - \frac{3}{\alpha\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^2} + 1 \tag{60}$$

$$s(z) = \frac{6\alpha\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right) - 4}{9\alpha^2\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)^3 - 6\alpha\left(1+W\left[\frac{\beta}{\alpha}\left(\frac{1}{1+z}\right)^{\frac{1}{\alpha}}\right]\right)} \tag{61}$$

From the graphical representations of the cosmological parameters in terms of redshift z we see that our model resembles the present universe with accelerated expansion at late times.

6 Conclusion

In this paper, we investigate a Bianchi type I universe in the framework of Brans-Dicke theory of gravity with time-dependent cosmological constant by using the hybrid expansion law of the average scale factor a given in Eq. (26). The constraints on the constants α and β can be obtained from observational data. Akarsu et al. [39], have obtained the values $\alpha = 0.488$ and $\beta = 0.444$ by using 580 SNIa data points [42] and $H(z)$ data from various sources [43–45]. The ranges of α and β are also obtained as $0.228 < \alpha < 1.09$ and $0.121 < \beta < 0.617$ by considering 580 SNIa and 25 $H(z)$ data points lying in the redshift ranges $0.015 < z < 1.414$ and $0.07 < z < 1.750$ respectively. In our study, in order to investigate the physical behaviours of the cosmological parameters, we choose $\alpha = 0.229$ and $\beta = 0.491$. In order to obtain a cosmological model with constraints from the observational data and in view of the equations (30) and (31) we take the values for the constants as $\alpha = 0.229$, $\beta = 0.491$, $m = 0.502$ and $\omega = 1.307$. The graphs for the cosmological parameters are drawn in terms of the cosmic time t and also in terms of the redshift z. We observe that

- The scale factor a and the scalar field ϕ increases with the cosmic time t.
- The Hubble parameter H decreases and gradually approaches zero with the increasing cosmic time t.
- The deceleration parameter q decreases with the increase in cosmic time t and tends to -1 at late times.
- The cosmological constant Λ keeps decreasing throughout the evolution of the universe and tends to zero as $t \to \infty$.
- The pressure p is negative and keeps decreasing throughout the evolution of the universe.
- The energy density ρ decreases initially and then increases with increasing cosmic time t.
- The EoS parameter η shows transitions from initially decreasing state to increasing state, then it decreases again and tends to -1 as the universe evolves. Throughout the transition the EoS parameter η takes the value $-1 < \eta < \frac{-1}{3}$ (quintessence phase), then $\eta = -1$ (ΛCDM model) and then $\eta < -1$ (phantom era). As the cosmic time t increases our universe enters into phantom era.
- The values of jerk parameter j and lerk parameter l fall initially but rise later and finally tend to 1 with the evolution of the universe.
- The value of snap parameter changes from initial negative values to positive ones with the increasing cosmic time t and then tends to 1.
- The increase in the spatial volume V throughout the cosmic evolution indicates the expansion of the universe.

• The expansion scalar θ decreases with the increasing cosmic time t.
• The shear scalar σ^2 and the mean anisotropy parameter A_m decrease and both tend to zero with the increase in the cosmic time t. This shows that the anisotropy dies out in the course of evolution resulting into an isotropic model at late times.
• The present value of the statefinder pair is $\{r = 1, s = 0\}$ which shows that our model resembles the ΛCDM model. It is in agreement with recent observational data.
• The SEC is violated at late times indicating the accelerated expansion of the universe.

From the physical behaviour of the various cosmological parameters, we observe that the constructed model might be helpful in studying the dynamics of the evolving universe as the findings are quite satisfactory with the current observational data.

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Bianchi Type-I Cosmological Model with a Special Law of Hubble Parameter in $f(R, T)$ Theory of Gravity

Chandra Rekha Mahanta and Kankana Pathak

Abstract In this chapter, we study a spatially homogeneous and anisotropic Bianchi type-I universe filled with perfect fluid within the framework of $f(R, T)$ theory of gravity for the functional form $f(R, T) = f_1(R) + f_2(T)$. We presume $f_1(R) = R$ and $f_2(T) = \lambda T$, where λ is a constant and obtain exact solution of the gravitational field equations by using a special law for the Hubble parameter H. The evolution of relevant cosmological parameters and the physical behaviour of the model are investigated. We also obtain the statefinder diagnostic pair and check the validity of the energy conditions. We find our results to be consistent with present observational data.

1 Introduction

Astrophysical and cosmological observations such as Supernovae Type Ia (SN Ia) [1, 2], Cosmic Microwave Background (CMB) [3–5], Wilkinson Microwave Anisotropy Probe (WMAP) [6–9], Large Scale Structure (LSS) [10], Baryon Acoustic Oscillation (BAO) [11, 12], and Galaxy redshift surveys strongly reveal and support that currently the universe is going through a phase of rapid expansion. In the literature, various theoretical frameworks have been put up to address this bizarre issue but the underlying reason for this increased growth in the cosmic expansion has not yet been identified. Within the context of General Relativity, the current phase of cosmic acceleration can be accommodated in the presence of the cosmological constant Λ, introduced by Einstein in his field equations. However, the cosmological constant suffers from some major problems such as the fine-tuning and the cosmic coincidence problems. Apart from the cosmological constant, there are typically two approaches

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in the literature to address the current acceleration in the expansion of the universe. One way is to modify the matter sector on the right-hand side of the Einstein field equations by including some new and hitherto unknown components of matter and energy, dubbed dark energy, which has a large negative pressure. The other way is to modify the geometric part on the left-hand side of the Einstein field equations. This can be done by modifying the gravitational Lagrangian part of the Einstein-Hilbert action in General Relativity as Einstein chose to define gravity as a manifestation of the curvature of space-time. A variety of modified theories of gravity such as $f(R)$ gravity [13], $f(R,T)$ gravity [14], $f(T)$ gravity [15], $f(G)$ gravity [16] and $f(Q)$ gravity [17] have been proposed in the literature as alternatives to the idea of dark energy. The simplest and the most straightforward modified theory of gravity is the $f(R)$ gravity in which the Ricci scalar R in the standard Einstein-Hilbert action is replaced by an arbitrary function $f(R)$ of (R) [13]. A further generalisation of $f(R)$ theory of gravity has been done by Harko et al. [14]. They have considered $f(R,T)$ modified theory of gravity in which the gravitational Lagrangian is given by an arbitrary function $f(R,T)$ of the Ricci scalar R and of the trace T of the stress-energy tensor. Using the metric formalism, Harko et al. derived the gravitational field equations and provided the field equations for specific models that correspond to some particular versions of the function $f(R,T)$. Additionally, they have derived the equation of motion of test particles and obtain the Newtonian limit in $f(R,T)$ theory of gravity. Subsequently, Houndjo [18] investigated the cosmological reconstruction in $f(R,T)$ gravity theory by attaching special attention to the functional form $f(R,T) = f_1(R) + f_2(T)$ and presented a model with transition of matter dominated phase of the universe to the accelerated phase. Since then the $f(R,T)$ theory of gravity has grown significantly in prominence in recent years, and several researchers have studied the $f(R,T)$ theory of gravity in various contexts as this theory addresses both the early time inflation and the late time cosmic acceleration, making it a strong contender to replace general relativity.

Chandel and Ram [19], Sahoo et al. [20], Pawar and Shahare [21] studied Bianchi type-III cosmological model in $f(R,T)$ theory of gravity. Chaubey and Shukla [22] investigated Bianchi types III, V and VI_0 models in $f(R,T)$ gravity. In recent times, multiple authors have looked into various features like dynamics of scalar perturbations, dynamical instability and expansion free conditions, evolution of axially symmetric anisotropic sources, bulk viscosity, wormhole, inflation, etc. in $f(R,T)$ theory of gravity [23–28]. Arora et al. [29] studied late-time viscous cosmology in $f(R,T)$ gravity. Bhattacharjee [30] studied inflation in mimetic $f(R,T)$ gravity. Bhatti et al. [31] studied stability of self-gravitating anisotropic fluids in $f(R,T)$ gravity. Singh et al. [32] investigated the dynamics of a parametrized dark energy model in $f(R,T)$ gravity. Koussour and Bennai [33, 34] studied Bianchi-I cosmological model using varying deceleration parameter and bulk viscosity in $f(R,T)$ theory of gravity. Vinutha and Kavya [35] have studied Bianchi type III, V and VI_0 cosmological models in $f(R,T)$ theory with quadratic functional form $R + \alpha R^2 + \lambda T$. Singh and Beesham [36–39] investigated LRS Bianchi type-I cosmological model in

$f(R, T)$ gravity. Many other authors have also studied Bianchi cosmological models in various other modified theories of gravity. Recently, Koussour et al. [40–43] investigated the anisotropy of the universe in $f(Q)$ gravity.

Spatially homogeneous and anisotropic Bianchi type space-times are considered in the literature to understand the evolution of the universe at its early phases and also at later phases as there is no observational evidence that rules out the possibility of an inhomogeneous or homogeneous and anisotropic phase although the current universe appears to be homogeneous and isotropic at a large scale. Experimental tests like Wilkinson Microwave Anisotropy Probe (WMAP) [7], Cosmic Microwave Background (CMB) anisotropy [5] etc. also suggest the presence of an anisotropic phase of the universe which have been wiped out in the course of cosmic evolution resulting into a homogeneous and isotropic universe. As a result, spatially homogeneous and anisotropic Bianchi type cosmological models have gained more popularity amongst the researchers. These models present a middle way between the completely inhomogeneous and anisotropic models and the homogeneous and isotropic models of the universe. Out of the Bianchi models I-IX, the simplest one is the Bianchi type-I model which evolves into a Friedmann-Lemaître-Robertson-Walker (FLRW) universe if filled with a perfect fluid. Many researchers have investigated Bianchi type universe filled with perfect fluid in different contexts [20, 33, 35, 44]

Motivated by the above discussion, in the current study, we take into account the Bianchi type-I cosmological model filled with perfect fluid in the framework of $f(R, T)$ theory of gravity with $f(R, T) = f_1(R) + f_2(T)$ where $f_1(R) = R$ and $f_2(T) = \lambda T$ and λ is a constant. The paper is organised as follows: In Sect. 2, we provide a basic formulation of $f(R, T)$ gravity theory and derive the gravitational field equations for the functional form $f(R, T) = R + \lambda T$. In Sect. 3, we obtain the field equations corresponding to the Bianchi type-I metric and in Sect. 4, exact solution of the field equations is obtained by taking into account a special law of the mean Hubble parameter. In Sect. 5, we discuss the physical behaviour of the model and in Sect. 6, we examine the statefinder diagnostic and energy conditions for this model. We conclude the paper with a brief discussion in Sect. 7.

2 Basic Formalism of $f(R, T)$ Gravity Theory

The action of $f(R, T)$ gravity theory was proposed by Harko et al. [14] as follows

$$S = \frac{1}{16\pi} \int f(R, T)\sqrt{-g}\, d^4x + \int L_m \sqrt{-g}\, d^4x \tag{1}$$

where $f(R, T)$ is an arbitrary function of the Ricci Scalar R and the trace T of the stress-energy tensor and L_m is the matter Lagrangian density.

The stress-energy tensor of matter is defined as [45]

$$T_{ij} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{ij}} \tag{2}$$

and the trace T is given by $T = g^{ij}T_{ij}$. By supposing that the metric tensor components g_{ij} are the only factors affecting the quantity L_m of matter and not on its derivatives, T_{ij} is obtained as

$$T_{ij} = g_{ij}L_m - 2\frac{\partial L_m}{\partial g^{ij}} \tag{3}$$

The field equation of $f(R, T)$ gravity is produced by taking variation of the action (1) with regard to the metric tensor components g^{ij} as

$$f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)f_R(R,T) = 8\pi T_{ij} - f_T(R,T)T_{ij} - f_T(R,T)\Theta_{ij} \tag{4}$$

where, $f_R(R,T) = \frac{\partial f(R,T)}{\partial R}$ and $f_T(R,T) = \frac{\partial f(R,T)}{\partial T}$ are the partial derivatives of $f(R,T)$ with respect to R and T respectively, ∇_i is the covariant derivative, $\Box = \nabla_k\nabla^k$ is the D'Alembert operator and Θ_{ij} is obtained by specifying the variation of T with respect to the metric tensor

$$\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{ij}} = T_{ij} + \Theta_{ij} \tag{5}$$

For a known matter Lagrangian L_m, Θ_{ij} can be calculated as

$$\Theta_{ij} = -2T_{ij} + g_{ij}L_m - 2g^{lk}\frac{\partial^2 L_m}{\partial g^{ij}\partial g^{lk}} \tag{6}$$

For a perfect fluid, the stress-energy tensor of matter is given by

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij} \tag{7}$$

where ρ is the energy density, p is the pressure and the four velocity u_i satisfies the conditions $u_i u^i = 1$ and $u^i \nabla_j u_i = 0$. The matter Lagrangian can be written as $L_m = -p$. Using Eq. (6), we get the expression for Θ_{ij} for the modification of stress energy tensor of perfect fluid as

$$\Theta_{ij} = -2T_{ij} - pg_{ij} \tag{8}$$

In general, the field equations are influenced by the physical character of the matter field by the tensor Θ_{ij}. As a result, in the situation of $f(R, T)$ gravity, based on the characteristics of the matter source, we have multiple theoretical models relating to each choice for the function $f(R, T)$

Assuming,

$$f(R, T) = f_1(R) + f_2(T) \tag{9}$$

where $f_1(R)$ and $f_2(T)$ are arbitrary functions of R and T respectively, from Eq. (4), the gravitational field equations of $f(R, T)$ gravity are obtained as

$$f_1^{'}(R)R_{ij} - \frac{1}{2}f_1(R)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)f_1^{'}(R) = 8\pi T_{ij} - f_2^{'}(T)T_{ij} - f_2^{'}(T)\Theta_{ij} + \frac{1}{2}f_2(T)g_{ij} \tag{10}$$

where the prime denotes differentiation with respect to the argument. If the matter source is perfect fluid then the field equations of $f(R, T)$ gravity becomes

$$f_1^{'}(R)R_{ij} - \frac{1}{2}f_1(R)g_{ij} = 8\pi T_{ij} + f_2^{'}(T)T_{ij} + \left[f_2^{'}(T)p + \frac{1}{2}f_2(T)\right]g_{ij} \tag{11}$$

For the choice of $f_1(R) = R$ and $f_2(T) = \lambda T$, where λ is a constant, the gravitational field equations of $f(R, T)$ gravity from Eq. (11) are obtained as

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij} + \lambda T_{ij} + \left[\lambda p + \frac{\lambda}{2}T\right]g_{ij} \tag{12}$$

3 The Metric and Field Equations

The spatially homogeneous and anisotropic Bianchi type-I metric is

$$ds^2 = dt^2 - (A^2dx^2 + B^2dy^2 + C^2dz^2) \tag{13}$$

where A, B and C are functions of cosmic time t only. Using comoving coordinates and Eqs. (6)–(8), with a specific selection of the function given in Eq. (12), the field equations for the metric (12) leads to

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{B}}{B}\frac{\dot{C}}{C}+\frac{\dot{C}}{C}\frac{\dot{A}}{A}=\left(8\pi+\frac{3\lambda}{2}\right)\rho-\frac{\lambda p}{2} \tag{14}$$

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}}{B}\frac{\dot{C}}{C}=-\left(8\pi+\frac{3\lambda}{2}\right)p+\frac{\lambda\rho}{2} \tag{15}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\frac{\dot{B}}{B}=-\left(8\pi+\frac{3\lambda}{2}\right)p+\frac{\lambda\rho}{2} \tag{16}$$

$$\frac{\ddot{C}}{C}+\frac{\ddot{A}}{A}+\frac{\dot{C}}{C}\frac{\dot{A}}{A}=-\left(8\pi+\frac{3\lambda}{2}\right)p+\frac{\lambda\rho}{2} \tag{17}$$

where differentiation with respect to t is denoted by an overhead dot.

4 Exact Solutions for the Case $f(R, T) = f_1(R) + f_2(T)$

From Eqs. (15)–(17), we get

$$\frac{\dot{A}}{A}-\frac{\dot{B}}{B}=\frac{k_1}{a^3} \tag{18}$$

$$\frac{\dot{B}}{B}-\frac{\dot{C}}{C}=\frac{k_2}{a^3} \tag{19}$$

$$\frac{\dot{C}}{C}-\frac{\dot{A}}{A}=\frac{k_3}{a^3} \tag{20}$$

where k_1, k_2 and k_3 are arbitrary constants.

From Eqs. (18)–(20), we get

$$\frac{A}{B}=c_1\exp\left(k_1\int\frac{dt}{a^3}\right) \tag{21}$$

$$\frac{B}{C}=c_2\exp\left(k_2\int\frac{dt}{a^3}\right) \tag{22}$$

$$\frac{C}{A}=c_3\exp\left(k_3\int\frac{dt}{a^3}\right) \tag{23}$$

where $a=(ABC)^{\frac{1}{3}}$ represents the average scale factor and c_1, c_2 and c_3 are constants of integration.

From (21)–(23), we get

$$A=m_1 a\exp\left(\frac{k_1-k_3}{3}\int\frac{dt}{a^3}\right) \tag{24}$$

$$B = m_2 a \exp\left(\frac{-2k_1 - k_3}{3}\int \frac{dt}{a^3}\right) \tag{25}$$

$$C = m_3 a \exp\left(\frac{k_1 + 2k_3}{3}\int \frac{dt}{a^3}\right) \tag{26}$$

where m_1, m_2 and m_3 are arbitrary constants.

Now, the directional Hubble parameters are

$$H_1 = H + \frac{k_1 - k_3}{3a^3}, H_2 = H + \frac{-2k_1 - k_3}{3a^3}, H_3 = H + \frac{k_1 + 2k_3}{3a^3}$$

Also, the Ricci scalar curvature is

$$\begin{aligned} R &= -2\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\dot{C}}{C}\frac{\dot{A}}{A}\right) \\ &= -\left(6\dot{H} + 12H^2 + \frac{2(k_1^2 + k_1k_3 + k_3^2)}{3a^6}\right) \end{aligned} \tag{27}$$

Then Eqs. (14)–(17) reduce to

$$H_1H_2 + H_2H_3 + H_3H_1 = \left(8\pi + \frac{3\lambda}{2}\right)\rho - \frac{\lambda p}{2}$$

$$\Rightarrow \left(8\pi + \frac{3\lambda}{2}\right)\rho - \frac{\lambda p}{2} = 3H^2 - \frac{k_1^2 + k_1k_3 + k_3^2}{3a^6} \tag{28}$$

$$\dot{H}_2 + H_2^2 + \dot{H}_3 + H_3^2 + H_2H_3 = -\left(8\pi + \frac{3\lambda}{2}\right)p + \frac{\lambda\rho}{2} \tag{29}$$

$$\dot{H}_1 + H_1^2 + \dot{H}_2 + H_2^2 + H_1H_2 = -\left(8\pi + \frac{3\lambda}{2}\right)p + \frac{\lambda\rho}{2} \tag{30}$$

$$\dot{H}_3 + H_3^2 + \dot{H}_1 + H_1^2 + H_3H_1 = -\left(8\pi + \frac{3\lambda}{2}\right)p + \frac{\lambda\rho}{2} \tag{31}$$

Adding Eqs. (29)–(31), we get

$$2\dot{H} + 3H^2 + \frac{(k_1^2 + k_1k_3 + k_3^2)}{3a^6} = -\left(8\pi + \frac{3\lambda}{2}\right)p + \frac{\lambda\rho}{2} \tag{32}$$

Now, there are five unknowns, i.e. A, B, C, p, and ρ and four field equations, so to obtain a complete solution, we need one more relation for which we take into account the following particular type of Hubble parameter [46–48].

$$H = n(a^{-m} + 1) \tag{33}$$

where $m, n > 0$. On integration,

$$a = a_0(e^{mnt} - 1)^{\frac{1}{m}}$$

where a_0 is a constant of integration.

Without compromising generality, we assume, $a_0 = 1$, then we get

$$a = (e^{mnt} - 1)^{\frac{1}{m}} \tag{34}$$

therefore,

$$H = \frac{\dot{a}}{a} = \frac{ne^{mnt}}{(e^{mnt} - 1)} \tag{35}$$

The deceleration parameter is

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = -1 + \frac{m}{e^{mnt}} \tag{36}$$

Using the expression $a(t) = \frac{1}{1+z}$, where z is the redshift, we can derive the time-redshift relation as follows:

$$t = \frac{1}{mn} \log\left[1 + \left(\frac{1}{1+z}\right)^2\right] \tag{37}$$

The Hubble parameter and deceleration parameter in terms of redshift leads to

$$H(z) = n[1 + (1+z)^m] \tag{38}$$

and,

$$q(z) = -1 + \frac{m(1+z)^m}{1 + (1+z)^m} \tag{39}$$

Using (29) in (20)–(22), we get

$$\begin{aligned} A &= m_1(e^{mnt} - 1)^{\frac{1}{m}} \exp\left(\frac{k_1 - k_3}{3} \int \frac{dt}{(e^{mnt} - 1)^{\frac{3}{m}}}\right) \\ &= m_1(e^{mnt} - 1)^{\frac{1}{m}} \exp\left(\frac{k_1 - k_3}{3} G(t)\right) \end{aligned} \tag{40}$$

where

$$G(t) = \int \frac{dt}{(e^{mnt} - 1)^{\frac{3}{m}}} \tag{41}$$

$$B = m_2(e^{mnt} - 1)^{\frac{1}{m}} \exp\left(\frac{-2k_1 - k_3}{3} G(t)\right) \tag{42}$$

$$C = m_3(e^{mnt} - 1)^{\frac{1}{m}} \exp\left(\frac{k_1 + 2k_3}{3} G(t)\right) \tag{43}$$

Therefore, the line element (13) becomes

$$ds^2 = dt^2 - m_1^2(e^{mnt} - 1)^{\frac{2}{m}} \exp\left(\frac{k_1 - k_3}{3} G(t)\right) dx^2 - m_2^2(e^{mnt} - 1)^{\frac{2}{m}}$$
$$\exp\left(\frac{-2k_1 - k_3}{3} G(t)\right) dy^2 - m_3^2(e^{mnt} - 1)^{\frac{2}{m}} \exp\left(\frac{k_1 + 2k_3}{3} G(t)\right) dz^2 \tag{44}$$

For this solution,

$$H_1 = \frac{\dot{A}}{A} = \frac{ne^{mnt}}{(e^{mnt} - 1)} + \frac{k_1 - k_3}{3(e^{mnt} - 1)^{\frac{3}{m}}} \tag{45}$$

$$H_2 = \frac{\dot{B}}{B} = \frac{ne^{mnt}}{(e^{mnt} - 1)} + \frac{(-2k_1 - k_3)}{3(e^{mnt} - 1)^{\frac{3}{m}}} \tag{46}$$

$$H_3 = \frac{\dot{C}}{C} = \frac{ne^{mnt}}{(e^{mnt} - 1)} + \frac{k_1 + 2k_3}{3(e^{mnt} - 1)^{\frac{3}{m}}} \tag{47}$$

5 Physical Behaviour of the Model

An anisotropic, spatially homogeneous, Bianchi type-I metric represented by the Eq. (44) has the following kinematic and physical characteristics:

The spatial volume

$$V = ABC = (e^{mnt} - 1)^{3/m} \tag{48}$$

The scalar expansion

$$\theta = \frac{3ne^{mnt}}{(e^{mnt} - 1)} \tag{49}$$

The shear scalar

$$\sigma^2 = \frac{(k_1^2 + k_1k_3 + k_3^2)}{3(e^{mnt} - 1)^{6/m}} \tag{50}$$

The mean anisotropy parameter

$$A_m = \frac{1}{3}\Sigma_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2k^2(e^{mnt}-1)^2}{9n^2e^{2mnt}(e^{mnt}-1)^{6/m}} \tag{51}$$

where $k^2 = k_1^2 + k_1k_3 + k_3^2$

Then from Eqs. (29) and (32) by using Eqs. (34) and (35), the energy density ρ and the pressure p are obtained as

$$\rho = \frac{1}{(8\pi+\lambda)(8\pi+2\lambda)}\left[\frac{3(8\pi+\lambda)n^2e^{2mnt}+\lambda mn^2e^{mnt}}{(e^{mnt}-1)^2} - \frac{(8\pi+2\lambda)k^2}{3(e^{mnt}-1)^{\frac{6}{m}}}\right] \tag{52}$$

$$p = \frac{-1}{(8\pi+\lambda)(8\pi+2\lambda)}\left[\frac{3(8\pi+\lambda)n^2e^{2mnt}-(16\pi+3\lambda)mn^2e^{mnt}}{(e^{mnt}-1)^2} + \frac{(8\pi+2\lambda)k^2}{3(e^{mnt}-1)^{\frac{6}{m}}}\right] \tag{53}$$

The average scale factor (a), the spatial volume (V), expansion scalar (θ) and shear scalar (σ^2) are all divergent at $t = 0$, as shown by Eqs. (34), (48), (49) and (50). The anisotropy parameter A_m is infinite at time $t = 0$.

We plot the graphs of some parameters in terms of cosmic time t (in Billion years) to study the evolution of our model. For this purpose, we choose appropriate values of the constants relating to recent observations. According to WMAP data, the current value of the Hubble parameter is around 0.07 $GYrs^{-1}$ [49] and the current value of the deceleration parameter is -0.81 ± 0.14 in the confidence level 1σ in ΛCDM model [50]. We can obtain a relationship between the parameters m and n using these data, which is $m = \frac{0.0231}{0.07-n}$. Consequently, for both n and m to be positive, n must be within the range $0 < n < 0.07$ [48] and then m is in the range $0.33 < m < \infty$.

The deceleration parameter defines how the universe has evolved; a positive deceleration parameter value ($q > 0$) indicates a decelerating universe, whereas a negative deceleration parameter value ($q < 0$) indicates an accelerating expanding universe.

Figures 1 and 2 show that our model matches the observed values of the Hubble parameter and the deceleration parameter for $n = 0.062$ and $m = 2.8875$. We utilise these values for n and m while plotting the graphs of other parameters. Again, in order to obtain more accurate results from our model, we select reasonable values of the constants λ and k as $\lambda = 0.3$ and $k = 0.01$ for the graphs of the energy density ρ, pressure p and energy conditions.

Figures 3 and 4 show that the energy density ρ has very large value in the beginning and decreasing with cosmic time t and the pressure is also a decreasing function of time t and attains negative values at late times.

The following graphs show the evolution of the Hubble parameter, the deceleration parameter, the energy density and the pressure in terms of redshift(z).

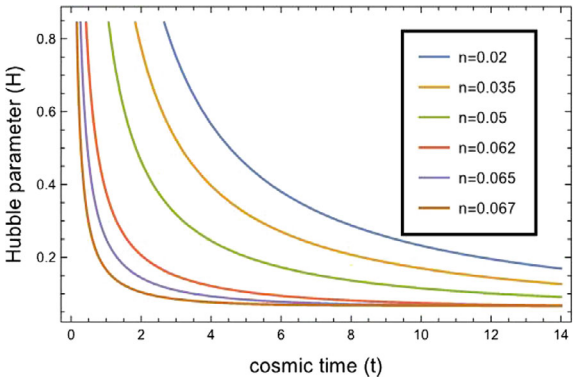

Fig. 1 Variation of the Hubble parameter H in terms of cosmic time t

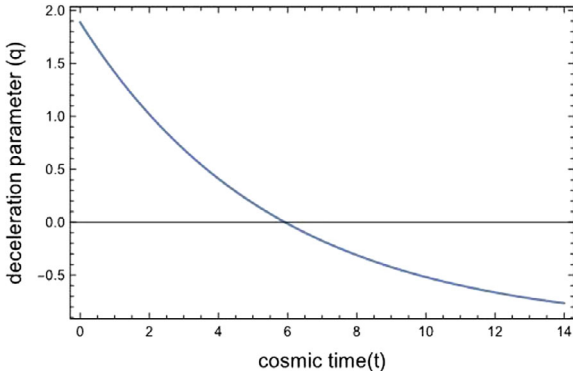

Fig. 2 The deceleration parameter q in terms of cosmic time t with $n = 0.062$

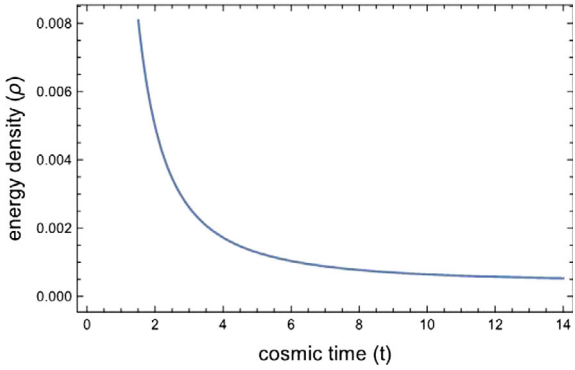

Fig. 3 The energy density ρ v/s cosmic time t with $n = 0.062$, $m = 2.8875$, $\lambda = 0.3$ and $k = 0.01$

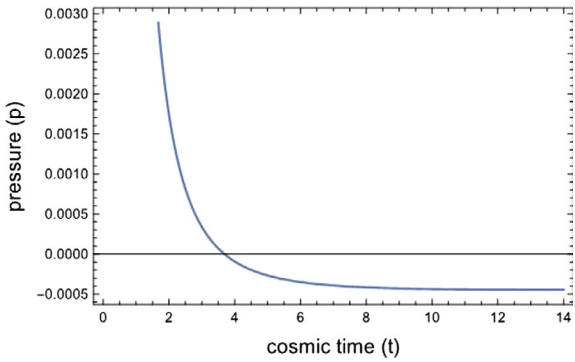

Fig. 4 The pressure p v/s cosmic time t with $n = 0.062$, $m = 2.8875$, $\lambda = 0.3$ and $k = 0.01$

From Figs. 5 and 7, we observe that the Hubble parameter (H) and the energy density (ρ) are increasing functions of redshift (z) and and tends to zero as $z \to -1$. From Figs. 6 and 8, we observe that the deceleration parameter (q) tends to –1 as redshift decreases and the pressure p is negative and increasing with the increase in redshift (z).

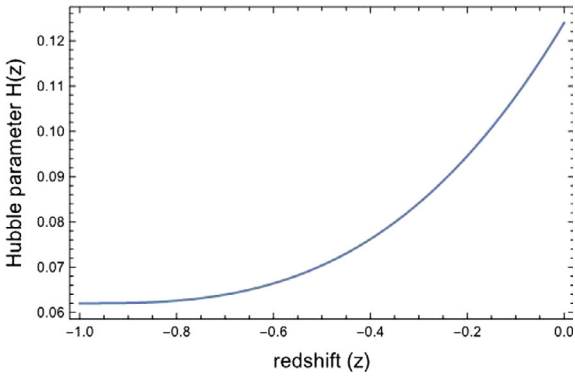

Fig. 5 The Hubble parameter $H(z)$ in terms of redshift z

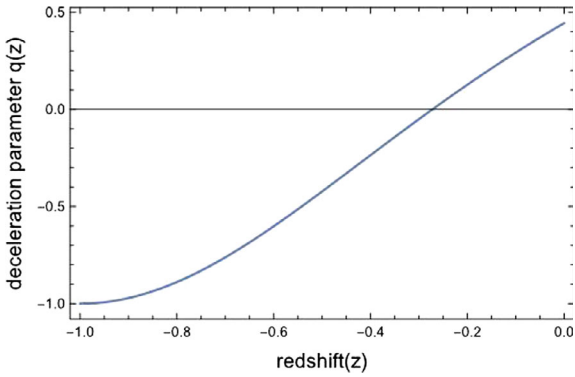

Fig. 6 The deceleration parameter $q(z)$ in terms of redshift z

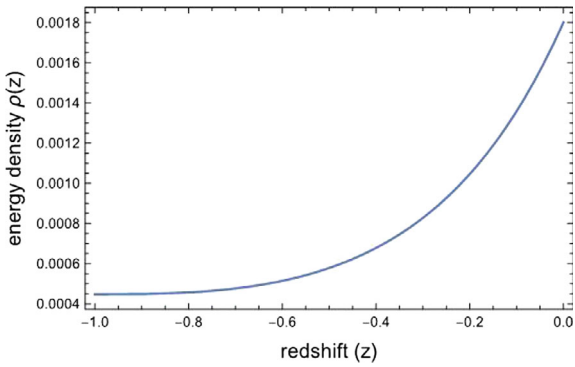

Fig. 7 The energy density $\rho(z)$ in terms of redshift z

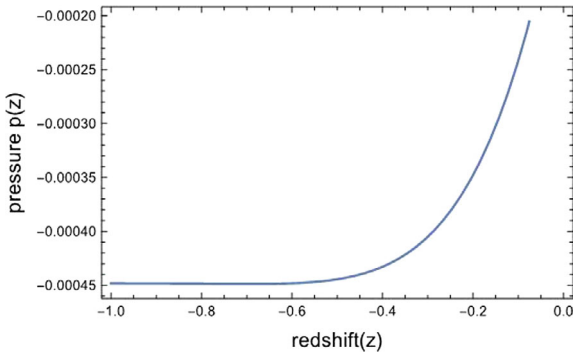

Fig. 8 The pressure $p(z)$ in terms of redshift z

Jerk, snap and lerk Parameters: The jerk parameter (j) is the third time derivative of the average scale factor a given by

$$j(t) = \frac{a^2}{\dot{a}^3}\frac{d^3a}{dt^3} = 1 + \frac{m^2 - 3m}{e^{mnt}} + \frac{m^2}{e^{2mnt}}$$

Similarly, the snap parameter (s) and the lerk parameter (l) are the fourth and fifth time derivatives of the average scale factor a respectively given by

$$s(t) = \frac{a^3}{\dot{a}^4}\frac{d^4a}{dt^4} = 1 + \frac{-m^3 + 4m^2 - 6m}{e^{mnt}} + \frac{7m^2 - 4m^3}{e^{2mnt}} - \frac{m^3}{e^{3mnt}}$$

$$l(t) = \frac{a^4}{\dot{a}^5}\frac{d^5a}{dt^5} = 1 + \frac{m^4 - 5m^3 + 10m^2 - 10m}{e^{mnt}} + \frac{11m^4 - 30m^3 + 25m^2}{e^{2mnt}} + \frac{11m^4 - 15m^3}{e^{3mnt}} + \frac{m^4}{e^{4mnt}}$$

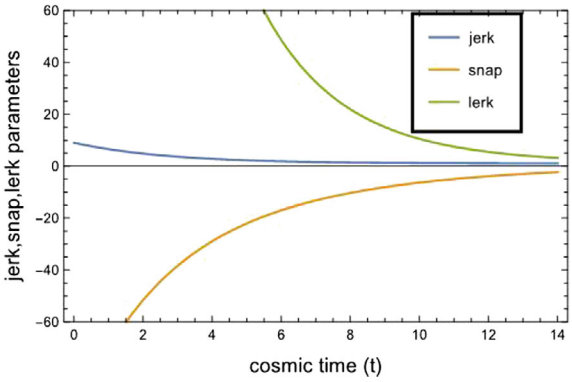

Fig. 9 Variation of jerk, snap and lerk parameter v/s cosmic time t

The jerk and lerk parameters exhibit diminishing tendencies, as seen in Fig. 9. The snap parameter has rising behaviour and is negative, both of which indicate a faster expansion, and both of these parameters are positive, denoting an accelerated expansion of the universe.

6 Statefinder Diagnostic and Energy Conditions

Statefinder parameter is a crucial geometrical diagnostic tool used to differentiate between several DE models, including ΛCDM (Lambda Cold Dark Matter), HDE (Holographic Dark Energy), CG (Chaplygin Gas), SCDM (Standard Cold Dark Matter) and Quintessence.

The statefinder pair $\{r, s\}$ [51] for our model are obtained as

$$r = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} = 1 - \frac{3m}{e^{mnt}} + \frac{m^2(e^{mnt}+1)}{e^{2mnt}}$$

$$s = \frac{r-1}{3\left(q-\frac{1}{2}\right)} \Rightarrow s = \frac{-6me^{mnt} + 2m^2(e^{mnt}+1)}{e^{mnt}(-9e^{mnt}+6m)}$$

For our model, the statefinder pair $\{r, s\}$ has a present value of $\{r, s\}= \{1.03, -0.0076\}$.

The energy conditions are characterised as

Weak Energy Condition (WEC): $\rho \geq 0$ and $\rho + p \geq 0$
Null Energy Condition (NEC): $\rho + p \geq 0$
Dominant Energy Condition (DEC): $\rho \pm p \geq 0$ and $\rho \geq 0$
Strong Energy Condition (SEC): $\rho + 3p \geq 0$ and $\rho + p \geq 0$

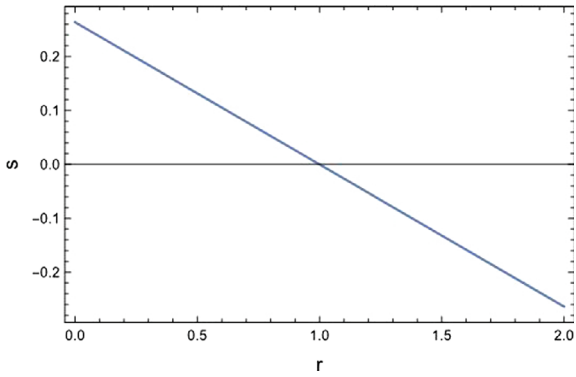

Fig. 10 Statefinder pair $\{r, s\}$

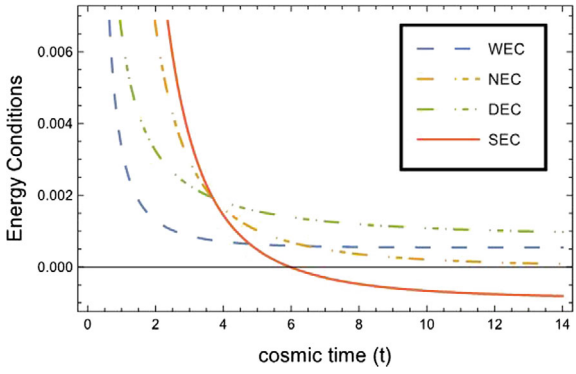

Fig. 11 The Energy conditions in terms of cosmic time t

From Fig. 10, we observe that in the beginning where $(r = 1, s = 0)$, it is clear that our model acts like the ΛCDM model. The model finally stays in the Chaplygin Gas area with $(r > 1, s < 0)$ after switching from ΛCDM to Chaplygin Gas [52].

Figure 11 shows an overview of the evolution of the energy conditions versus cosmic time t. While SEC is interrupted towards the end of the cosmic expansion, WEC, NEC and DEC are clearly fulfilled throughout the cosmic evolution.

7 Conclusion

We construct a cosmological model by assuming the mean Hubble parameter H to follow a special law $H = n(a^{-m} + 1)$ and study the physical behaviour of the model. For this purpose, we draw graphs of some cosmological parameters by choosing appropriate values of the constants m and n by keeping an eye on the current values of the Hubble parameter and the deceleration parameter from WMAP data. For our constructed model, we observe that:

- The Hubble parameter has a high value at cosmic time $t = 0$ and decreases throughout the evolution of the universe as shown in Fig. 1. Also, in case of redshift (z), the Hubble parameter is increasing with the increase in redshift (z).
- The energy density ρ decreases as cosmic time t increases, moving closer to zero at late time.
- The pressure p is declining with respect to cosmic time t and turns negative at late times indicating the transition from decelerated phase to late time accelerated phase.
- The anisotropy dies out in the course of the evolution and the model eventually attains isotropy in future.
- The evolution of the universe is depicted by the graphical representation of the deceleration parameter q for the model, which goes from the early deceleration phase to the current acceleration phase. Additionally, as seen in Fig. 6, the model enters an accelerating phase at low redshift values.
- The present value of the statefinder pair $\{r, s\}$, depicts switching from early ΛCDM model to the late Chaplygin Gas model.
- While the SEC is violated at a later stage in the cosmic history, the energy criteria (WEC, NEC and DEC) are fully satisfied.

Thus, we find that our findings concur with recent observational data.

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Mathematical Modeling and Analysis of COVID-19: A New Trajectory in the Model Framework

Sheralu Vadeo and Biju Kumar Dutta

Abstract The dawn of corona virus (COVID-19) has become a worldwide matter of concern. Some mathematical modeling has been developed to comprehend the spread of the virus and inhibit the virus infection. In order to make predictions, develop control strategies, and assess them, it is essential to have a better understanding of the transmission mechanism and the undermining elements of the spread of infectious diseases. Practically, mathematical modeling is a way for evaluating epidemics and the effects of various assumptions made about potential treatments. This study intends to acquire a more accurate mathematical model dynamics of the disease using computational modeling and compare it to the current model. This research incorporates a new trajectory in the study of COVID-19. The newly proposed model integrates quarantined and vaccinated data which has not been studied so far in my understanding. We derive the numerical approximations of the model equations employing computational modeling. As a result, the estimated parameters and initial population model states have different model dynamics. The findings in this analysis mark a significant advancement in the ability to forecast model dynamics for development capacities, interventions, and healthcare methodologies.

1 Introduction

The first corona virus was discovered in the 1930s which caused a bronchial infection in chickens. In the 1960s, SARS-CoV (Severe Acute Respiratory Syndrome Coronavirus) was firstly identified in humans [10]. According to a new study by a global team of virus scientists, raccoon dogs DNA was discovered in genetic samples taken at Wuhan market close to the site of the first corona virus case in human, supporting the idea that it may have originated in animals [2]. A collection of pneumonia cases was reported in Wuhan, China in 2019. The World Health Organization (WHO) for-

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mally disclosed the corona virus a worldwide epidemic on March 11, 2020 [15]. This virus may have originated from the seafood market and transported globally with returning travelers as the main carriers [6]. As recognized by WHO, the formulation of the mathematical models plays a vital role with public health decisions and strategy producers with proof-based measurements. The world recorded 678, 773, 973 confirmed cases, 6, 791, 680 deaths and 651, 506, 905 recovered cases on February 21st, 2023 as per [18].

The findings can be further enhanced by developing a proposed model based on the law of mass action employing transfer variables and highlighting some important model components. The model introduced in this paper gives attention to important history factors primarily according to sensitivity analysis and arithmetic outcomes more precisely wider.

The major purpose of research regarding the transmission of epidemics is understanding the mechanism of infectious disease transmission and the factors that contribute to its spread so that predictions can be made, control strategies can be developed, and these strategies can be evaluated [17]. An evaluation tool that helps conduct virtual epidemics and examine the effects of various treatments assumptions is mathematical modeling [9]. This study intends to acquire a more accurate mathematical model dynamics of the disease using computational modeling and compare it to the current model.

This research contributes to our knowledge of the importance of socioeconomic interactions in the propagation of infections among a range of methods. By integrating some more realistic elements that are not accounted for in the SIR model (Susceptible Infected Removed), which has generated disease diffusion models dating back to Kermack and McKendrick [11], the research contributes to the epidemiology literature.

In another study, Khoshnaw [13] took into account the recovered individual, the reported symptomatic rate of death, and the unreported symptomatic rate of death in the methodology for forecasting of COVID-19. In computational simulation, the constructed model comprises of six parameters that may be examined. As they become asymptomatically exposed at the rate of change $\beta S(I + U)$, susceptible people (S) shift in to the infected category (I). Later on, the asymptomatic group (I), with transition rate γI either become reported symptomatic infected individuals (W) or move to the unreported symptomatic category (U) at the rate δI. Groups W and U transfer through to the recovered groups at transition rates of ηW and ηU. Furthermore, several other infected people who are both unreported and reported could eventually die with $\alpha_1 U$ and $\alpha_2 W$ respectively.

Below are the rates of the model reactions:

$$S \xrightarrow{v_1} I, I \xrightarrow{v_2} W, I \xrightarrow{v_3} U, W \xrightarrow{v_4} R, U \xrightarrow{v_5} R, W \xrightarrow{v_6} \text{Death}, U \xrightarrow{v_7} \text{Death},$$

where

$$v_1 = \beta S(I + U), v_2 = \gamma I, v_3 = \delta I, v_4 = \eta W, v_5 = \eta U, v_6 = \alpha_1 U, v_7 = \alpha_2 W.$$

The model dynamics are given below

$$\begin{aligned}
\frac{dS}{dt} &= \beta S(I+U),\\
\frac{dI}{dt} &= \beta S(I+U) - (\gamma+\delta+\alpha_1)I\\
\frac{dU}{dt} &= (\delta-\alpha_1)I - (\eta+\alpha_2)U\\
\frac{dW}{dt} &= (\gamma-\alpha_1)I - (\eta+\alpha_2)W\\
\frac{dR}{dt} &= \eta(W+U),
\end{aligned}$$

with $S(0) = S_0 > 0, I(0) = I_0 > 0, U(0) = U_0 > 0, W(0) = W_0 > 0, R(0) = R_0 \geq 0$.

Diseases are universal and some continue to claim millions of lives worldwide. Daniel Bernoulli, who researched on the effects of cow-pox inoculation on the transmission of small pox in 1760, was the first to apply mathematical modeling to epidemic propagation in [3]. In their paper Hufnagel et al. [7] constructed a probabilistic model based on mathematical modeling (SIR) to explain the widespread of epidemics including Severe Acute Respiratory Syndrome (SARS). For ninety days after the original outbreak, the approach was applied in 2003 to foresee the development of SARS in Hong Kong. The simulation result was strikingly similar to the spread of SARS as reported by World Health Organization (WHO). They also stated that based on the findings, the model might predict the development of future infectious diseases and identify vulnerable places ahead of time.

According to Liang [7], SARS, MERS (Middle East Respiratory Syndrome) and COVID-19 have all been linked to transmission laws. He did a comparison study between COVID-19, SARS and MERS. By examining the rate of expansion and ongoing control of the antimicrobial infection, he built transmission growth models. Nonlinear fitting is used to obtain the variables needed for this conceptual framework. They discovered that COVID-19 rate of increase is about twice as fast as SARS and MERS, with a fold change of two to three days. This suggests that, absent human intervention, the majority of people with COVID-19 will increase by twofold in two to three days. In 2001, David and Lang [16] deduced the SIR model for infectious illness transmission in New York, such as the Hong Kong flu. The researchers assumed a fully homogeneous population with no age spatial or social structure throughout the flu epidemic by omitting births, immigrations, and deaths due to non-flu causes. They discovered that if the disease contact number is limited enough, no epidemic can form and vice versa. Khoshnaw et al. [13] discovered some fundamental set of variables that significantly influence how quickly the virus spread among the model compartments. Afzal et al. [1] discussed the most notable characteristics, applications, constraints, and future outlook of the most recent mathematical models of

COVID-19 which can help make mathematical models that take into account the present problems with viral infections even better. Two approaches were put forth in this regard to address two issues related with COVID-19. As the initial problem with short range estimates for COVID-19 epidemic, Asif studies recommended an alternate strategy which included Auto-Regressive Integrated Moving Average (ARIMA) and Wavelet Based Forecasting Model (WBF) models.

2 Preliminaries

Based on the models disclosed, mathematical models are used to analyze the emergence, evolution and prognosis of COVID-19. At various levels, a variety of mathematical models are used to predict/forecast the disease, the function of quarantine, deaths, reported and unreported cases and so on. The Table 1 shows a brief COVID-19 study of various mathematical modeling that are used [1].

Table 1 Various mathematical models used for modeling of COVID-19

Model	COVID-19 studies
Ordinary and Partial Differential Equations (ODE and PDE)	Simulation of disease transmission, infection and death
Susceptible-Infected-Recovered (SIR)	Data fitting, management techniques and tracking transmission and recovery rates over time
Susceptible-Infected-Quarantined-Recovered (SIQR)	Management techniques, quarantine
Susceptible-Infected-Symptomatic Unreported-Symptomatic reported-Recovered (SIUWR)	Modeling the transmission rate, spread and simulation modeling to help the estimation and prediction of infected individuals

SIR compartmental frameworks are frequently used to model epidemic dynamics owing to their intuitive and quantitative clarity. Table 2 summarizes the research methodology, benefits, and drawbacks of COVID-19 statistical models [1].

Table 2 An overview of COVID-19 mathematical modeling

Models for every category	What does it research?	Advantages	Disadvantages
Deterministic compartmental model (SIR)	Data fitting, management techniques and tracking transmission and recovery rates over time	Easy to use and understand	Other COVID-19-related parameters are not mentioned
Deterministic compartmental model (SIR)	Data fitting, management techniques and tracking transmission and recovery rates over time	Easy to use and understand	Other COVID-19-related parameters are not mentioned
Susceptible-Infected-Quarantined-Recovered (SIQR)	Management techniques, quarantine	Provides accurate case prediction and improves when expanded to include more states and factors	Modeling is difficult since it requires making assumptions
Susceptible-Infected-Symptomatic Unreported-Symptomatic Reported-Recovered (SIUWR)	Modeling the transmission rate, spread and simulation modeling to help the estimation and prediction of infected individuals	Provides a significant advancement in anticipating future model dynamics for development programmes, treatments and health care methods	Modeling is complicated and to improve interventions and health care programmes, some mathematical techniques are required

3 Methods

When modeling physiological processes with model equations, the notion of chemical kinetic theory is an effective technique. Model states, parameters and equations are key assumptions for building such models [12]. We take into account n irreversible chemical equations and m chemical components with the following stochastic equations.

$$\sum_{j=1}^{m} \alpha_{ij} x_j \xrightarrow{k_i} \sum_{j=1}^{m} \beta_{ij} x_j, \; i = 1, 2, \ldots, n, \tag{1}$$

where, $x_j, \; j = 1, 2, \ldots, m$ are chemical elements, α_{ij} and β_{ij} are stochastic variables; $k_i > 0, \; i = 1, 2, \ldots, n$ are reaction constants.

Using mass law action

$$v_i = k_i \prod_{j=1}^{m} x_j^{\alpha_{ij}}(t),\ i = 1, 2, \ldots, n. \tag{2}$$

Hence, the model dynamics for the proposed model in this study can be formed by using

$$\frac{dx}{dt} = \sum_{i \in J \subset \mathbb{R}} \pi_i v_i \tag{3}$$

with $x_i(0) = x_i^0$, where $\pi_i = \beta_{ij} - \alpha_{ij}$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. It is possible to write (3) as

$$\frac{dx_j}{dt} = h_j(x, k), \tag{4}$$

where $x \in \mathbb{R}^m$ and $k \in \mathbb{R}^n$. The model input is assumed to be a vector of parameters, and the model output is a vector of model states (variables). The modification in model values $x_j,\ j = 1, 2, \ldots, m$ with respect to model parameters $k_p,\ p = 1, 2, \ldots, n$ is referred to as local sensitivity. Sensitivity equation is

$$s_{ip} = \frac{\partial x_j}{\partial t} = \lim_{\Delta k_p \to 0} \frac{x_j(k_p + \Delta k_p) - x_j(k_p)}{\Delta k_p}. \tag{5}$$

In most cases, direct sensitivity analysis is employed to calculate the sensitivity equation, which involves solving differential equations for sensitivity coefficients.

$$\frac{\partial s_{ip}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial k_p}\right) = \frac{\partial}{\partial k_p}\left(\frac{\partial x_i}{\partial t}\right) = \frac{\partial}{\partial k_p}\left(h_j(x(t), k)\right). \tag{6}$$

Equation (6) may then be differentiated by chain rule and Jacobian matrix is formed from the local sensitivity equation

$$S' = H_{k_p} + J \cdot S,\ \ p = 1, 2, \ldots, n. \tag{7}$$

The matrices $S,\ H_{k_p}$ and I are defined as

$$S = \begin{pmatrix} \frac{\partial x_1}{\partial k_p} \\ \frac{\partial x_2}{\partial k_p} \\ \vdots \\ \frac{\partial x_m}{\partial k_p} \end{pmatrix}, H_{k_p} = \begin{pmatrix} \frac{\partial x_1}{\partial k_p} \\ \frac{\partial x_2}{\partial k_p} \\ \vdots \\ \frac{\partial x_m}{\partial k_p} \end{pmatrix}, J = \begin{pmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_m} \end{pmatrix}.$$

The initial state of the system (7) is governed by the given input value k_p and the initial state of the independent variables x_j.

3.1 Mathematical Modeling for Coronavirus

In context of a graphical network, such illustration clarifies the epidemic models [5, 14]. The baseline populations and interaction parameters for the model are primarily based on confirmed cases in China. We expand on the earlier model described in [13] in this study. Figure 1 displays the model diagram with the transition rates among the seven compartments. $G = (N, L)$, G is the model graph, $N = (S, V, I, U, W, Q, R)$ is the collection of the state variable and $L = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12})$ is the collection of links.

Examining the human population as a composite of seven smaller categories:

1. Susceptible group(S): This category represents a person in good health.
2. Vaccinated group (V): This category represents the population who becomes protected against a disease through vaccination.
3. Asymptomatic individuals(I): This category represents the population that does not display any symptoms, yet they are capable of infecting the susceptible group by droplets or direct contact.
4. Unreported symptomatic category(U): It depicts a person who displayed symptoms consistent with COVID-19, but is not recognized as a COVID-19 suspect by the authorities.
5. Reported symptomatic group(W): This group depicts a person who contracts the disease, develops COVID-19 symptoms, and is identified by the government through a quick test or deliberately reporting to the hospital.
6. Quarantined group(Q): This group represents those infected individuals who are made to isolate from the susceptible population.
7. Recovered group(R): This category consists of infected individuals who have recovered from the virus.

As seen in Fig. 1, twelve parameters have been generated in this model, which may be examined via computational simulations. Susceptible people(S) either shifts into infected people(I) at the transition rate $\beta S(I + U)$ when they are asymptomatic infected or they shift into the vaccinated category(V) at the transition rate of $\sigma_1 S$ and further move into the asymptomatic infected category(I) with the transition rate of $\sigma_2 V$. The asymptomatic category(I) furthermore develops into reported symptomatic category(W) with a transition rate of γI or to the unreported symptomatic category(U) for rate of change of δI. In addition, some individual from the asymptomatic infected compartment may die with transition rate $\alpha_1 I$. We then extend the model by introducing the quarantined category(Q) with the rate of transition νI after which the population moves into the recovered group with transition rate of ηQ. With the transition rates ηW and ηU both groups W and U migrate into the recovered category(R). Furthermore, with transition rates of $\alpha_2 U$ and $\alpha_3 W$, some individuals from the unreported and reported populations may die.

The following are the rates of the reaction

$$S \xrightarrow{v_1} I, S \xrightarrow{v_2} V, V \xrightarrow{v_3} I, I \xrightarrow{v_4} \text{Death}, I \xrightarrow{v_5} W, I \xrightarrow{v_6} Q,$$
$$I \xrightarrow{v_7} U, W \xrightarrow{v_8} R, Q \xrightarrow{v_9} R, U \xrightarrow{v_{10}} R, U \xrightarrow{v_{11}} \text{Death}, W \xrightarrow{v_{12}} \text{Death},$$

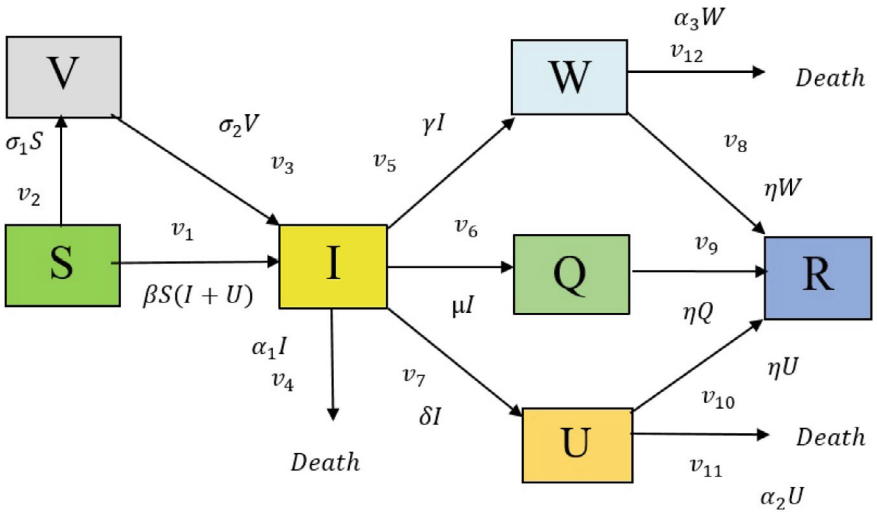

Fig. 1 The model diagram with transition rates

where $v_1 = \beta S(I+U)$, $v_2 = \sigma_1 S$, $v_3 = \sigma_2 V$, $v_4 = \alpha_1 I$, $v_5 = \gamma I$, $v_6 = \mu I$, $v_7 = \delta I$, $v_8 = \eta W$, $v_9 = \eta Q$, $v_{10} = \eta U$, $v_{11} = \alpha_2 U$, $v_{12} = \alpha_3 W$.
Using Eqs. (2)–(5), the model dynamics are

$$\begin{aligned}
\frac{dS}{dt} &= -\beta S(I+U), \\
\frac{dV}{dt} &= \sigma_1 S - \sigma_2 V, \\
\frac{dI}{dt} &= \beta S(I+U) + \sigma_2 V - (\gamma + \delta + \alpha_1 + \mu) I \\
\frac{dU}{dt} &= (\gamma - \alpha_1) I - (\eta + \alpha_2) U \\
\frac{dW}{dt} &= (\gamma - \alpha_1) I - (\eta + \alpha_3) W \\
\frac{dQ}{dt} &= \mu I - \eta Q, \\
\frac{dR}{dt} &= \eta(W + U + Q),
\end{aligned} \tag{8}$$

with $S(0) = S_0 > 0, V(0) = V_0 > 0, I(0) = I_0 > 0, U(0) = U_0 > 0, W(0) = W_0 > 0, Q(0) = Q_0 > 0, R(0) = R_0 \geq 0$.

The (8) also describes the model constant variables and initial states and definitions.

3.2 Computational Results

Since an epidemic model has a lot of components, mathematical models are used to support and render a more thorough and straightforward approach to studying infectious diseases. Hypothetical research aids in predicting future system dynamics and locating important model parameters. The provided mathematical models are practical resources for assessment and suggestions for improved virus management and prevention in the case of COVID-19.

The WHO situation report included the variables and the initial populations values used in this investigation. Using SCILAB 6.1.0 on Windows 10, 64-bit machine, we determine the model iterative solutions to the Eq. (8) for different indicators and initial populations. The Table 3 for COVID-19 is the confirmed data in China presented in [13] and some data were estimated for computational purpose in the study of this model.

Table 3 The initial populations and parameters for the breakout of COVID-19 in China, with biological definitions

Symbols	Biological definitions	Estimated values
$S(0)$	Initial susceptible population	11.081×10^6
$V(0)$	Initial vaccinated population	0
$I(0)$	Initial asymptomatic population	3.62
$U(0)$	Initial symptomatic unreported population	0.2
$W(0)$	Initial vaccinated population	4.13
$Q(0)$	Initial quarantined population	0
$R(0)$	Initial recovered population	0
β	The rate of change between susceptible and asymptomatic people	4.44×10^{-8}
γ	The rate of change of asymptomatic to reported symptomatic group	0.1142
δ	The rate of change of asymptomatic to unreported symptomatic group	0.0285
μ	The rate of change of asymptomatic to quarantined group	0.0540
$\frac{1}{\eta}$	The average duration of symptomatic infectious disease	$\frac{1}{7}$
α_1	Asymptomatic death rate	0
α_2	Death rate for unreported symptomatic	1.5×10^{-4}
α_3	Death rate for reported symptomatic	1.782×10^{-6}
σ_1	The rate of change between susceptible and vaccinated group	0.4×10^{-4}
σ_2	The rate of change between vaccinated to asymptomatic infected compartment	0.001×10^{-4}

The characteristics of susceptible, vaccinated, asymptomatic, symptomatic reported, and symptomatic unreported, quarantined and recovered categories are displayed in Fig. 2. Throughout time of 40 days, the susceptible individuals eventually decline and remains constant, and the variations of vaccinated and recovered category goes up. Between a period of 40 to 80 days, the population of infected person in the asymptomatic, symptomatic and quarantined groups has abruptly changed.

Figure 3 explain how the asymptomatic infected individuals affect the other compartments. It is evident that the relations on the unreported infected, reported infected and quarantined individuals are comparatively same. Whereas, the dynamical relation for susceptible and recovered individuals differ faintly.

The transition rate δ for asymptomatic, vaccinated, symptomatic and quarantined categories is shown in Fig. 4. The influence of this variable may clearly be seen on V, I, U, W and Q. We can see that if the value rises from 0.0285 to 0.6 the population of symptomatic unreported category grows (see Fig. 4c). On the other hand, when the transition rate δ decreases, the vaccinated, asymptomatic, reported symptomatic infected and quarantined groups increases, as seen in Fig. 4a, b, d and e.

Figure 5 illustrates the instances of asymptomatic, vaccinated, reported symptomatic infected, unreported symptomatic infected and quarantined compartments affected by the transition rate η . The outcome of this parameter may clearly be seen on I, V, U, Q and W. As γ is increased, it is evident that the characteristics of these population flattens.

Figure 6 shows η influence on the symptomatic unreported, reported symptomatic, quarantined and recovered compartments. As η value drops, the population of unreported, reported and quarantined people grows notably (see Fig. 6a and b). The population of recovered category expands substantially and quickly stabilises as the value of η rises. An alternative key stage in parameter analysis (8) is applying the notion of local sensitivity analysis as indicated in (7). Using SCILAB, we can determine the local sensitivity of the system in terms of dependent variables for the mathematical formulation. For instance, it indicates that extremely sensitive model factors in the spread of this epidemic are the rate of communication among asymptomatic and symptomatic unreported as well as the norm duration infected people experience symptoms. The mortality rate of asymptomatic, unreported symptomatic, reported symptomatic, and rate of change between susceptible, vaccinated and asymptomatic infected patient sensitivity to model characteristics. Therefore, focusing on the described methodologies for the modified COVID-19 framework, certain discoveries and analytical conclusions are projected. The summary of the above results is

1. SCILAB is used to estimate the dynamics of the system for susceptible, vaccinated asymptomatic, both unreported and reported, quarantined and recovered persons. The association among infected categories and the other groups is apparent in the framework of COVID-19. The association of the dynamics is identical for unreported, reported symptomatic and quarantined stages, while the system characteristics for susceptible, vaccinated and recovered categories are altered.

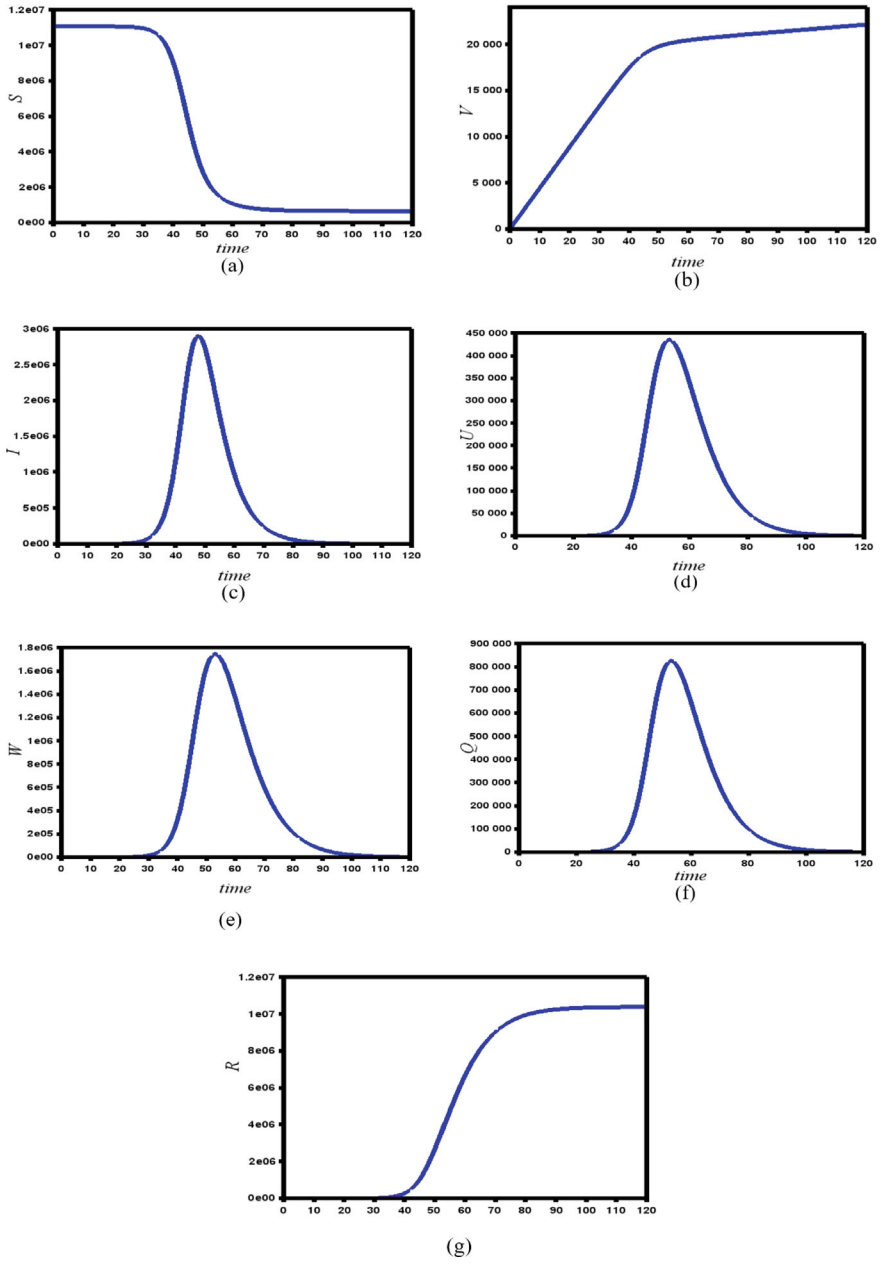

Fig. 2 SCILAB-based algorithmic analysis for the variables **a** susceptible, **b** vaccinated, **c** asymptomatic, **d** unreported symptomatic, **e** reported symptomatic, **f** quarantined, **g** recovered individuals; provided in the COVID-19 system

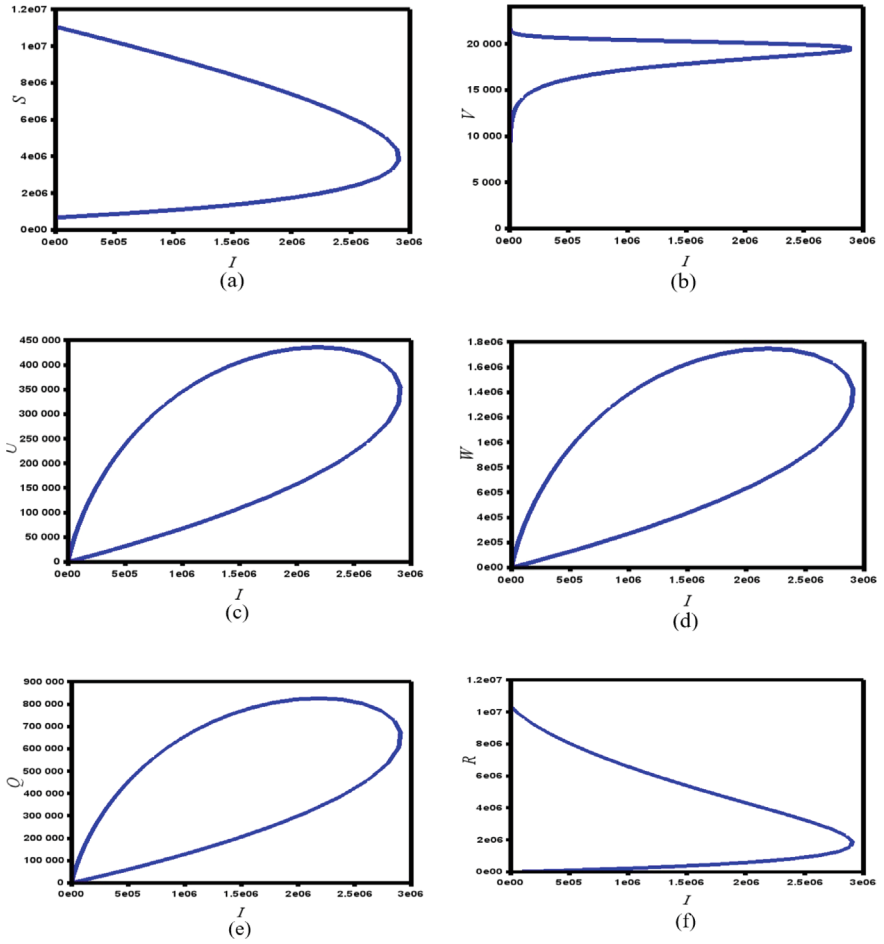

Fig. 3 Computational analysis using SCILAB to study the asymptomatic infected individual effect on **a** susceptible, **b** vaccinated, **c** symptomatic unreported, **d** symptomatic reported, **e** quarantined, and **f** recovered categories

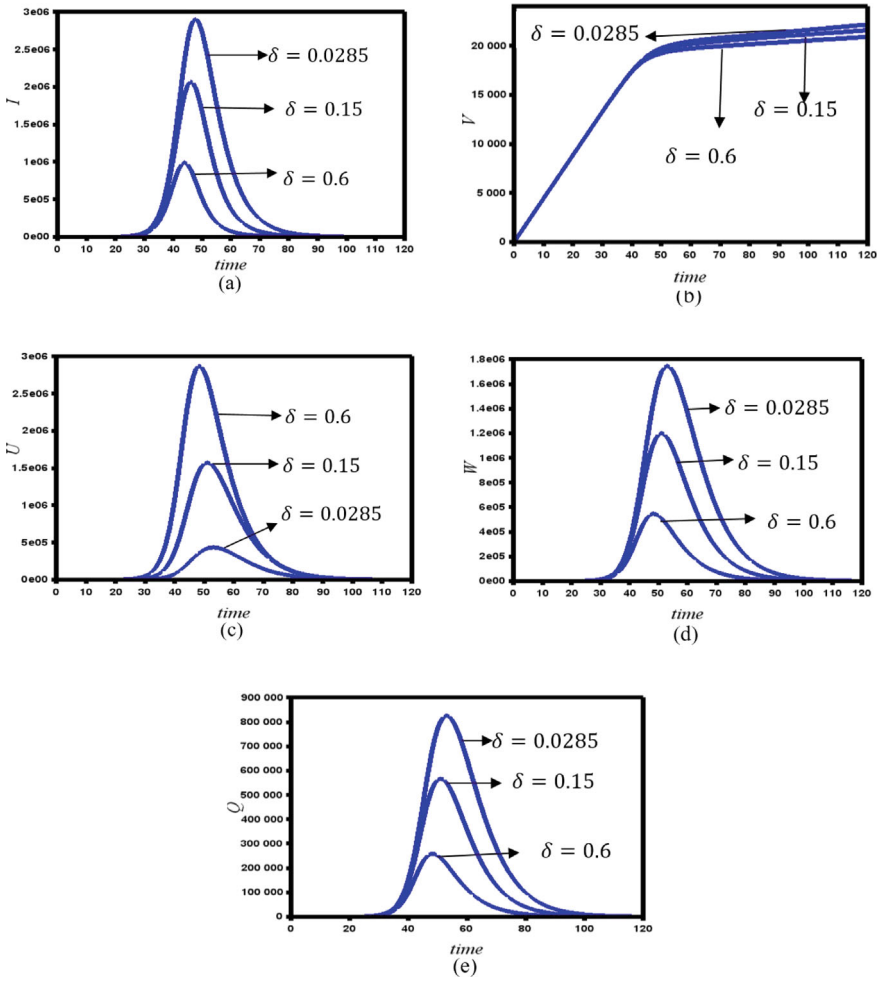

Fig. 4 Computational analysis using SCILAB to study the effect of rate of change $\delta = 0.0285, 0.15, 0.6$, on **a** asymptomatic, **b** vaccinated, **c** symptomatic unreported, **d** symptomatic reported, and **e** quarantined categories

2. Vaccinated, asymptomatic, symptomatic reported, symptomatic unreported and quarantined groups affected by transition rate δ are explained. This variable function well in V, I, U, Q and W.

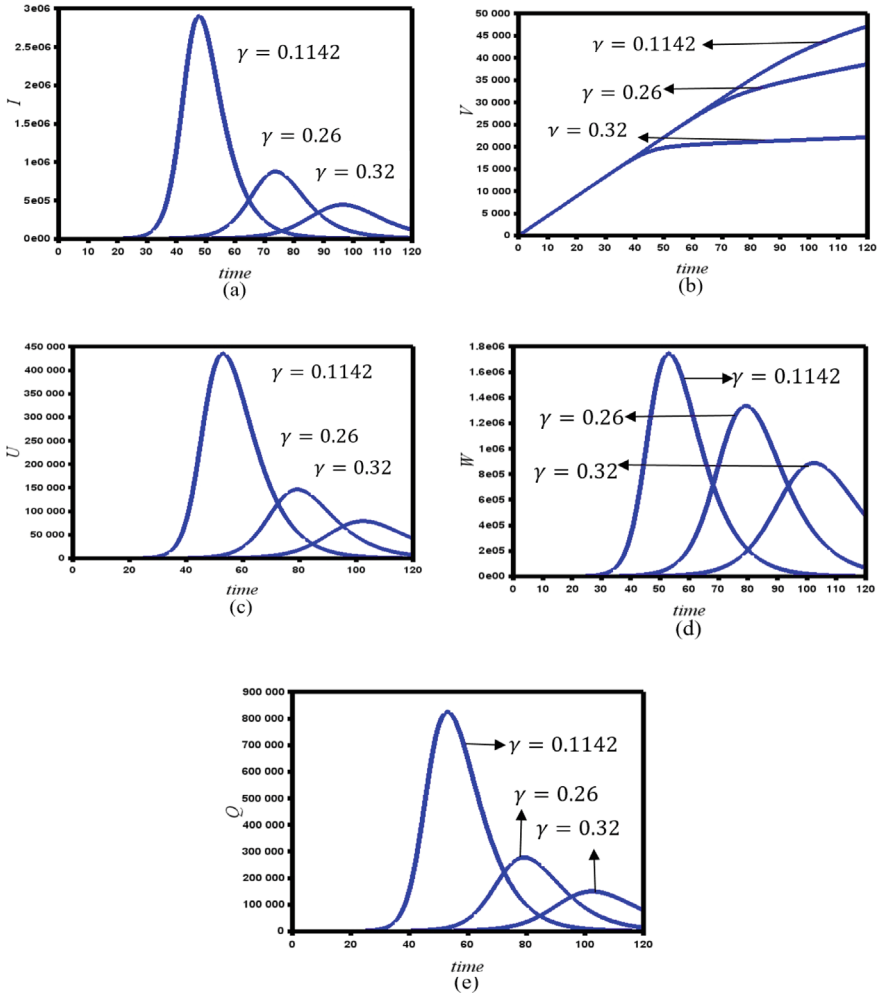

Fig. 5 Computational analysis using SCILAB to study the effect of rate of change $\gamma = 0.1142, 0.26, 0.32$, on **a** asymptomatic, **b** vaccinated, **c** symptomatic unreported, **d** symptomatic reported, and **e** quarantined categories

3. Vaccinated, asymptomatic, symptomatic reported, symptomatic unreported and quarantined person were impacted by the rate of change γ. As γ is increased, it is evident that the population flattens which is a critical component to manage the infection.
4. While η value lowers, the population of reported, unreported and quarantined category increases drastically, meanwhile the population of recovered category expands substantially and quickly stabilises as the value of η rises.

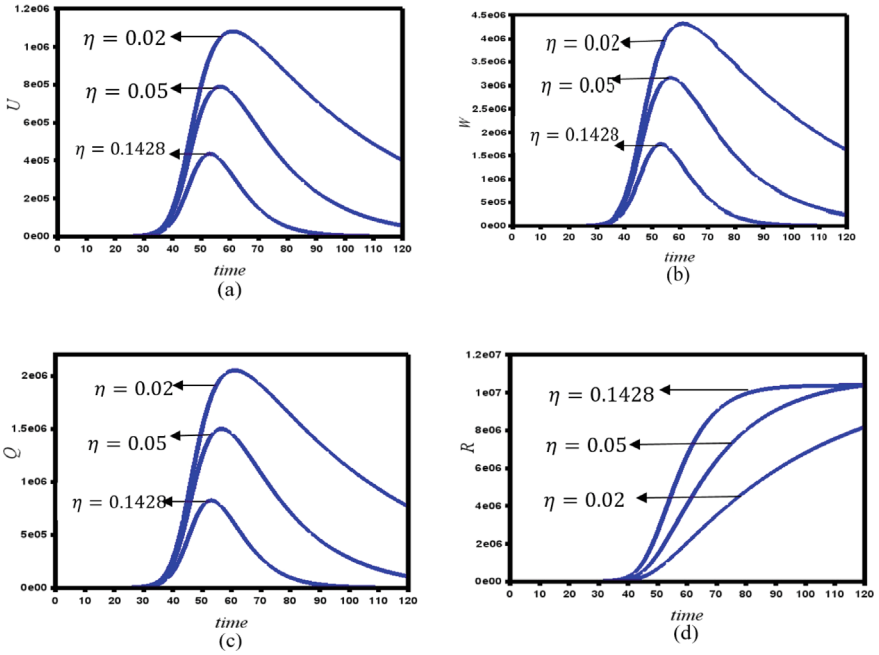

Fig. 6 Computational analysis using SCILAB to study the effect of $\eta = 0.1428,\ 0.05,\ 0.02$ on **a** symptomatic unreported, **b** symptomatic reported, **c** quarantined and **d** recovered compartments

3.3 A Quantitative and Qualitative Analysis of COVID-19 in India

The source of data of COVID-19 confirmed asymptomatic infected cases of India on 2nd August, 2021 has been considered for the computational purpose for this model. Approximately 29.1 million of India's population received full COVID-19 vaccination as of May 4, 2021 presented in [4]. The computational results are primarily obtained from [8] to study the COVID-19 cases in India and some transition rates were assumed for experimental purpose, as shown in Table 4.

The characteristics of susceptible, vaccinated, asymptomatic, symptomatic reported, and symptomatic unreported, quarantined and recovered groups are depicted in Fig. 7. Following 60 days, the susceptible populations progressively start to decline and remain stable, also the characteristics of vaccinated and recovered groups increases with time. Between 20 to 80 days, the infection rate in the asymptomatic, symptomatic and quarantined groups changes considerably.

The outcome of the asymptomatic infected on the other compartments is explained in Fig. 8. We can see that the relations on the unreported infected, reported infected and quarantined individuals are similar. The model dynamics for the susceptible, vaccinated and recovered individuals differs to a slight extent.

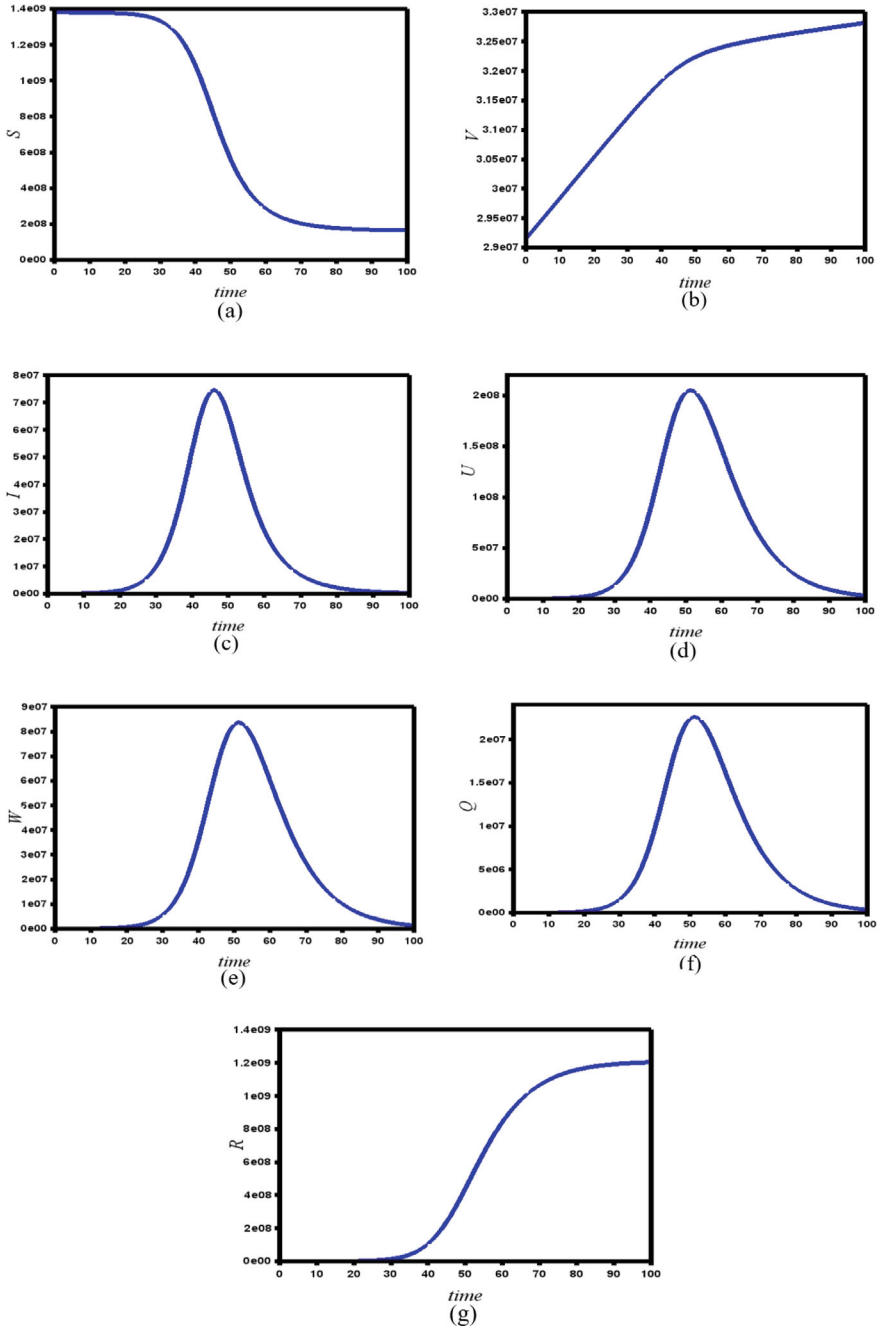

Fig. 7 SCILAB-based algorithmic analysis of the variables **a** susceptible, **b** vaccinated, **c** asymptomatic, **d** symptomatic unreported, **e** symptomatic reported, **f** quarantined, **g** recovered categories; provided in COVID-19 system (8)

Table 4 The initial populations and parameters for the breakout of COVID-19 virus in India, with biological definitions

Symbols	Biological definitions	Estimated values
$S(0)$	Initial susceptible population	1.38×10^9
$V(0)$	Initial vaccinated population	2.915×10^7
$I(0)$	Initial asymptomatic population	40709
$U(0)$	Initial symptomatic unreported population	0.3
$W(0)$	Initial vaccinated population	4.00
$Q(0)$	Initial quarantined population	0.1
$R(0)$	Initial recovered population	0
β	The rate of change between susceptible and asymptomatic people	2.95×10^{-9}
γ	Rate of change of asymptomatic to reported symptomatic	0.20
δ	Rate of change of asymptomatic to symptomatic unreported	0.491
μ	Rate of change of asymptomatic to quarantined	0.0540
$\frac{1}{\eta}$	The average duration of symptomatic infectious disease	$\frac{1}{7}$
α_1	The rate of death of asymptomatic population	0
α_2	The death rate for unreported symptomatic	2.51×10^{-4}
α_3	The death rate for reported symptomatic	1.822×10^{-5}
σ_1	The transmission rate between susceptible and vaccinated people	0.5×10^{-4}
σ_2	The rate of change among vaccinated and asymptomatic people	0.008×10^{-5}

Figure 9 illustrates the rate of change δ influencing asymptomatic, vaccinated, symptomatic reported, unreported symptomatic and quarantined compartments. The compartments I, V, U, Q and W are clearly affected by this parameter. For instance, increasing δ from 0.0285 to 0.6 significantly increases the population of unreported symptomatic category (see Fig. 9c). On the other hand, when the transition rate δ decreases, the asymptomatic, vaccinated, reported symptomatic and quarantined groups increases, as seen in Fig. 9a, b, d and e.

In Fig. 10, the characteristics of asymptomatic, symptomatic reported, and symptomatic unreported categories affected by the rate of change γ is portrayed. The compartments I, V, U, W and Q is evidently impacted by the parameter. When the value of γ is increased, it may be demonstrated that the characteristics for such conditions flattens.

The impact of η on symptomatic unreported, symptomatic reported, quarantined and recovered groups is shown in Fig. 11. As η drops, the population of unreported, reported and quarantined categories goes up dramatically (see Fig. 11a, b and c). The

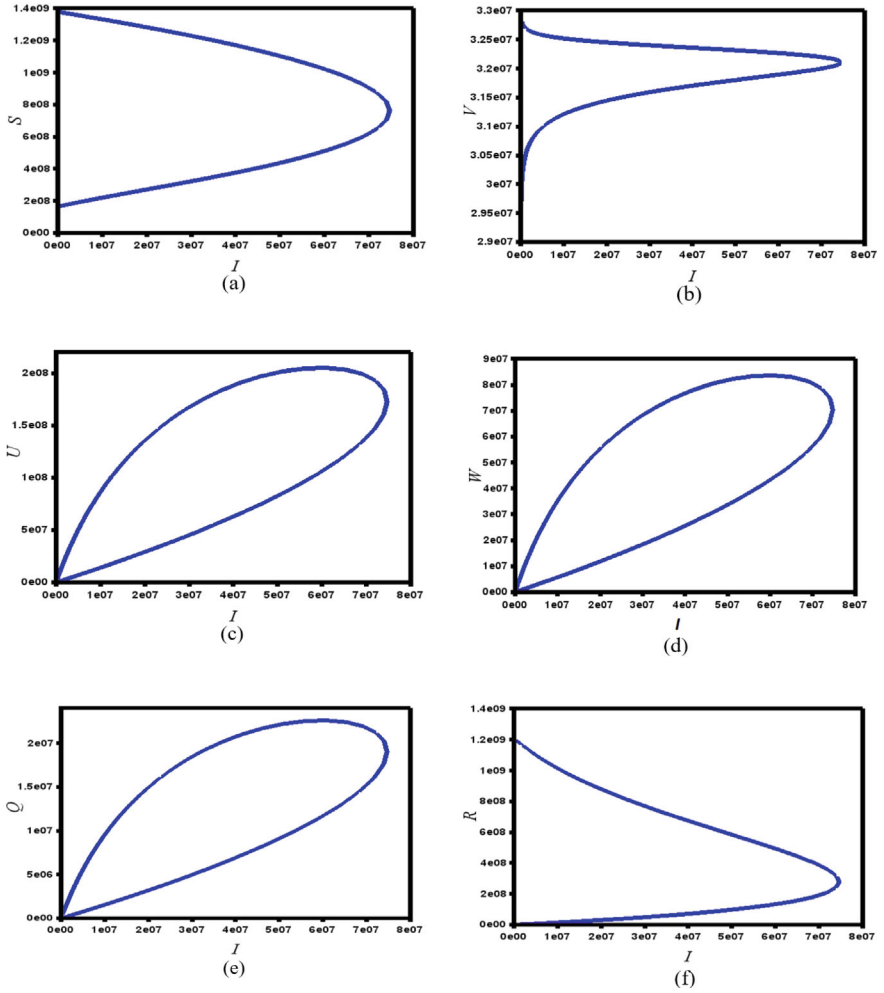

Fig. 8 Computational analysis using SCILAB to study the effect of asymptomatic infected on **a** susceptible, **b** vaccinated, **c** symptomatic unreported, **d** symptomatic reported, **e** quarantined, and **f** recovered categories

population of recovered group rise substantially and become extremely steady as η tends to rise.

The summary of the above results is:

1. In the COVID-19 model, it is demonstrated how affected patients relate to the rest of the categories. For reported, unreported symptomatic and quarantined stages, the correlation remains intact, however the characteristics for susceptible, vaccinated and recovered populations tends to vary.

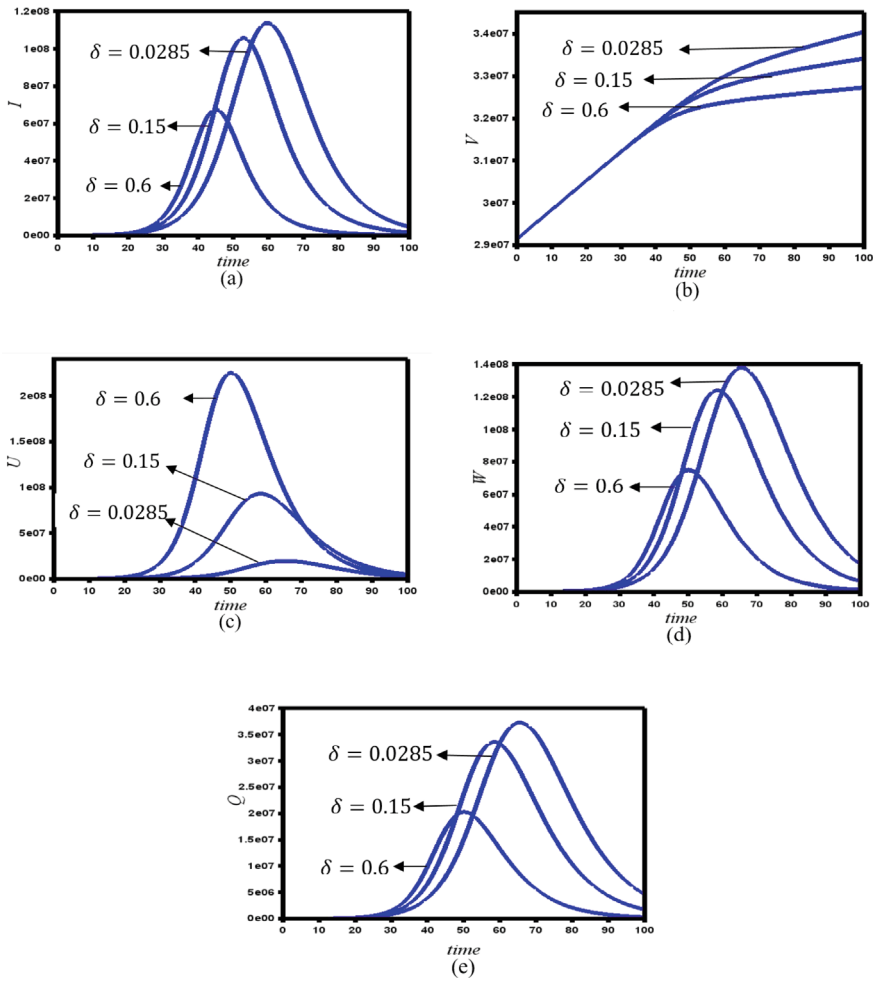

Fig. 9 Computational analysis using SCILAB to study the rate of change $\delta = 0.0285,\ 0.15,\ 0.6$ on **a** asymptomatic, **b** vaccinated, **c** symptomatic unreported, **d** symptomatic reported, and **e** quarantined groups

2. Asymptomatic, vaccinated, reported symptomatic, unreported symptomatic and quarantined groups affected by transition rate δ are explained. The compartments $I,\ V,\ U,\ Q$ and W are evidently impacted by the parameter.
3. Asymptomatic, vaccinated, reported symptomatic, unreported symptomatic and quarantined categories has been adversely affected by γ. When the value of γ is increased, it can be shown that the characteristics for such conditions flattens.
4. As η value drops, the population of both unreported, reported and quarantined categories goes up significantly, whereas if η value grows, the population of recovered compartment surges and settles.

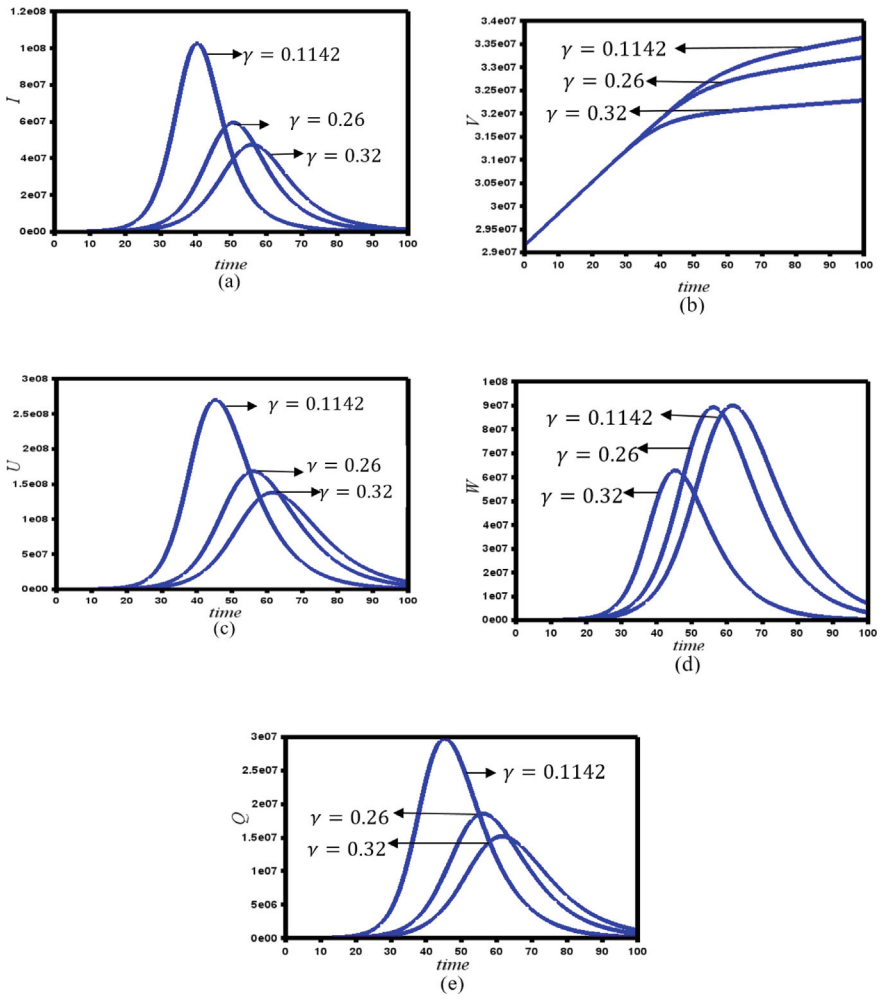

Fig. 10 Computational analysis using SCILAB to study the effect of $\gamma = 0.1142,\ 0.26,\ 0.32$, on **a** asymptomatic, **b** vaccinated, **c** symptomatic unreported, **d** symptomatic reported, and **e** quarantined groups

4 Conclusion

Many simulation techniques have been examined for potential enhancements and produce estimates for different parameters. Identifying essential crucial model parameters is a topic that can be researched further. This enables international attempts to give controversial subjects more attention. The postulated corona virus disease model is extensive, needing the application of mathematical approaches to improve actions and healthcare systems.

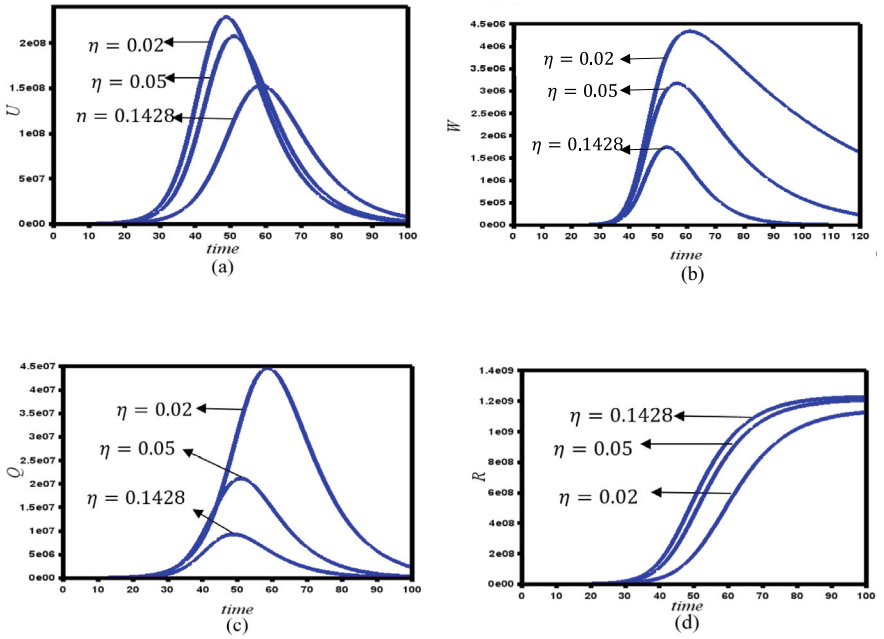

Fig. 11 Computational analysis using SCILAB to study the effect of $\eta = 0.1428,\ 0.05,\ 0.02$ on **a** symptomatic unreported, **b** symptomatic reported, **c** quarantined and **d** recovered groups

Based on the computational results using SCILAB 6.1.0 on Windows 10 64-bit machine for different populations and settings, the model utilized in this paper has been developed further. Some key findings can aid in a more comprehensive and effective comprehension of the proposed model. We find certain important crucial characteristics that play a significant influence in the transmission of the virus throughout the model classes using computational modelling. This is a significant development in how we perceive COVID-19 and the way it expands aggressively. In this work, we looked into a few more crucial model parameters. The rate of change in asymptomatic and reported groups, for example, has a big impact on the model states dynamics. Furthermore, this research provides estimations and projections for the number of infected people. As a result, our findings in terms of identifying important parameters are in line with clinical and biological findings.

As a result, new ideas and actions for regulating COVID-19 is suggestible based on how the parameter is impacted over the model conditions. In comparison of the newly modified model with a study proposed by Khoshnaw [13], we conclude that the new model gives a better projection of the disease which can aid the health-care strategies for any interventions or immunization campaigns. As a result, healthcare professionals should pay closer attention to quarantine facilities in order to better control the disease. To prevent infections from being passed from one person to another by interacting exposed areas, it is strongly recommended that quarantined areas remain

segregated and utilize only their own belongings, bathrooms, and personal items. A theoretical approach is that the infected population is effectively reduced by limiting direct interaction with asymptomatic-symptomatic and healthy populations. We can conclude that one mechanism must be devised to place asymptomatic infected individuals in quarantine as soon as possible.

Future study into identifying important critical parts could broaden the scope of the new COVID-19's explanations. Future studies should explore into further potential rate of change among the compartments. For example, adding transition route across symptomatic unreported and symptomatic reported will improve the model.

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Numerical Modelling in Heat Transfer Analysis of SWCNT/EG Nanofluid in a Stretching Sheet: A Darcy–Forchheimer Model

Bikash Das and Sahin Ahmed

Abstract The prime objective of the investigation is to study the effect of *SWCNT* nanoparticles in the base fluid Ethylene glycol on the hydrodynamics and heat transmission through a stretching surface in a Darcy–Forchheimer porous regime. The novelty of the investigation is to examine the behaviour of the heat conduction due to superior thermal conductivity of *SWCNT* under the applicability of magnetic fluid, Forchheimer force, wall suction and injection and the presence of nanoparticles in the basefluid moving along a stretching sheet. Non-linear partial differential equations are transformed to ODE using appropriate similarity transformations and a finite difference scheme of bvp4c has been adopted to obtain the solutions of various physical variables in the form of curves via MATLAB codes. Velocity distribution, nanofluid temperature and molar species along with engineering key attributes—Skin friction, Nusselt number, Sherwood number are plotted with the variation of controlling parameters of the nanofluid. The implication of this study has been accentuating in the field of bio-engineering and thermal manufacturing engineering with heat transfer analysis in nanomaterials.

Keywords Bvp4c · *SWCNT* · Wall suction and injection · Forchheimer · Non-Darcy Lorentz force

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1 Introduction

Modern equipment and technology are vital for our existence. We are improving them in a variety of methods to make them more effective. So, an innovative method was discovered by Choi and Eastman [1] to add nanosized (diameter $\leq$ 100 nm) particles into base fluids to improve the process of machines to cool. When compared to conventional fluids, the dispersed particles, which are typically metallic or metallic oxides, provide better heat transmission characteristics, and it has been proven that adding nanoparticles to a base fluid improves the thermal conductivity of the resulting nanofluid. The improved heat transmission properties of nanofluids have made them useful in a variety of heat exchange processes with a variety of commercial applications, including food pasteurization, fuel cells, biomedicine/microelectronics and cancer therapy. Nanofluids have been used in numerous uses, including power plant fuel reduction, car cooling and nanodrug delivery. According to Wong and De Leon [2], using nanofluidic coolants in electronics saves energy and lowers pollution, resulting in reduced production costs. Pandey and Kumar [3] studied the motion of nanofluid through an elongated cylinder while taking radiation and natural convection into account. Bhattacharyya et al. [4] investigated the influence of composite nanostructures in nanofluids and used a statistical technique to evaluate the properties of copper and graphene type nanoparticles. The impact of thermal radiation due to heat transport, mechanism of slip flow and wall suction on non-steady nanofluid with Lorentz force towards the wall on a stretched surface was investigated by Sreedevi et al. [5]. Notably, the authors [6–9] have scrutinized the boundary layer flow in nanofluid across stretched sheets (linear/non-linear) under various physical circumstances. The application of the Coriolis force on the dynamics of a Carreau–Yasuda spinning nanofluid subjected to the Darcy–Forchheimer model was investigated by [10]. The Significance of the magnetic field, Darcy–Forchheimer law, Brownian motion and non-Fourier heat flux model on the dynamics of Casson dusty fluid has been studied by [11]. Recently, authors [12–14] have studied the influence of Lorentz force on the flow behaviour of different nanofluids. The influence of thermal radiation on the MHD flow of casson hybrid nanofluid over a vertical plate was scrutinized by [15], while in [16] studied the influence of Hall current on the motion of rotating fluid under the influence of Lorentz force. The researcher [17–19] has studied the influence of various physical phenomena on heat and mass transfer by natural convection over stretching surface.

CNTs are formed with thin-shaped cylindrical structures with a diameter of nanoscale that have applications in a variety of disciplines including energy, biomedical, electronics, the environment and healthcare. Iijima [20] was the first to describe carbon nanotubes (CNTs). A non-Darcy flow model of *CNT-Water* nanofluid under the influence of Lorentz force was presented by Upreti et al. [21]. Gautam et al. [22] examined the effects of the Forchheimer medium's molar flux rate of reaction, thermal energy and Lorentz force on the fluid of nanoparticles travelling along a stretched wall in non-Darcy materials. The *MWCNT* and *SWCNT* have been designed by Khan et al. [23] to move along a surface in a manner similar to a normal fluid in

order to analyse the improved heat transfer caused by the increase in thermal growth in nanofluid. Influence of Lorentz force on the convective flow of nanofluid with nano-inclusion of motile micro-organism was investigated by Das and Ahmed [24].

Through this investigation, an attempt has been made to study the MHD Darcy–Forchheimer flow of *SWCNT-EG* nanofluid with the application of suction/injection at the surface. The main focus of this investigation is to study the impact of different parameters on velocity, temperature and concentration profiles. The set of governing equations is first transformed into non-dimensional ODEs using appropriate similarity transformations, and solution of the ODEs is plotted with the help of the MATLAB bvp4c solver. The Influence of various parameters on skin friction and the Nusselt number have also been plotted along with velocity, temperature and concentration profiles.

2 Formulation of the Problem

In this study, a 2-D laminar, steady and incompressible flow of *SWCNT/EG* nanofluid over a flat stretching surface embedded in a porous medium in presence of magnetic field of intensity B_0 acting normal to the surface has been considered. The contribution of heat source/sink, ohmic heating and viscous dissipation are included in the energy equation. A first-order chemical reaction is also taken into consideration in mass diffusion equation. The velocity of the stretching sheet is taken to be $U_w(x) = cx$, $(c > 0)$ and fluid motion is aligned to the x-axis. Furthermore, it is considered that the surface is being heated by a nanofluid with a heat exchange coefficient h_f and a temperature T_w. Temperature at free stream is supposed to be T_∞. The thermophysical properties of base fluid and nanoparticles are obtainable in Table 1. Under the above conventions, the resulting governing equations are given by [25, 26]. The flow configuration has been presented in Fig. 1.

The assumptions made throughout the analysis are as follows (Table 2):

- The flow Newtonian, laminar, incompressible and steady;
- Size and shape of the nanoparticles are considered to be uniform;
- The nanofluid is electrically conducting;
- The porous medium is uniform permeability and thermally isotropic. The local thermal equilibrium conditions between porous medium and nanofluid exists;

Table 1 Thermophysical properties of base fluid and nanoparticles

Physical properties	ρ	C_p	κ	β	σ
Ethylene Glycol (EG)	1114	2415	0.252	57×10^{-5}	1.67×10^{-6}
Cu	8933	385	401	1.67×10^{-5}	5.95×10^{6}
SWCNT	2600	425	6600	1.9×10^{-5}	10^{7}

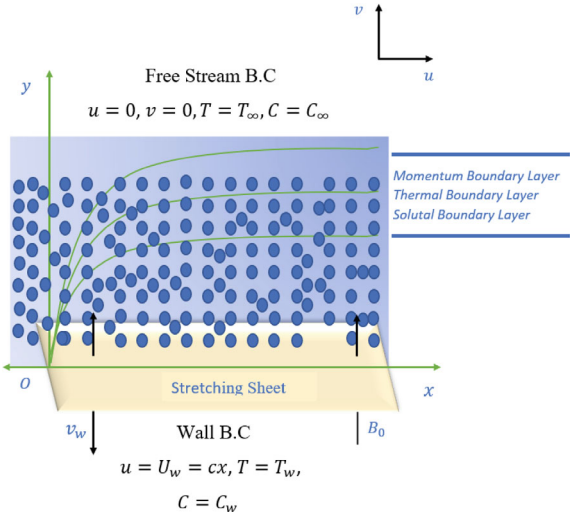

Fig. 1 Schematic diagram of the flow model

Table 2 Thermophysical properties of nanofluid [28–30]

Physical properties	Nanofluid
Viscosity	$\mu_{nf} = \mu_f (1-\varphi)^{-2.5}$
Density	$\rho_{nf} = (1-\phi)\rho_{bf} + \phi\rho_{np}$
Thermal expansion coefficient	$(\rho\beta)_{nf} = (\rho\beta)_{bf}(1-\varphi) + \varphi\frac{(\rho\beta)_{np}}{(\rho\beta)_{bf}}$
Electrical conductivity	$\sigma_{nf} = \sigma_{bf}\left[1 + \frac{3\left(\frac{\sigma_{np}}{\sigma_{bf}} - 1\right)\varphi}{\left(\frac{\sigma_{np}}{\sigma_{bf}} + 2\right) - \left(\frac{\sigma_{np}}{\sigma_{bf}} - 1\right)\varphi}\right]$
Thermal conductivity	$\kappa_{nf} = \kappa_{bf}\left(\frac{\kappa_{np} + \kappa_{bf} - \varphi(\kappa_{bf} - \kappa_{np})}{\kappa_{np} + \kappa_{bf} + \varphi(\kappa_{bf} - \kappa_{np})}\right)$
Heat capacity	$(\rho C_p)_{nf} = (\rho C_p)_{bf}(1-\varphi) + \varphi(\rho C_p)_{np}$
Volume fraction	$\varphi = \frac{\frac{W_{np}}{\rho_{np}}}{\frac{W_{np}}{\rho_{np}} + \frac{W_{bf}}{\rho_{bf}}}$

- Darcy–Forchheimer model is employed for momentum analysis in the porous medium;
- Magnetic Reynolds number is taken to be small so that the induced magnetic field can be neglected;
- There is no slip between the fluid and the stretching surface;

The thermophysical properties of nanofluid is given in (Table 2).
Based on the above consideration, the governing equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_{nf}\frac{\partial^2 u}{\partial y^2} - \frac{\nu_{nf}}{k_{fh}}u - \sigma_{nf}B_0^2 u - Fu^2, \tag{2}$$

$$(\rho C_p)_{nf}u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa_{nf}\frac{\partial^2 T}{\partial y^2} + \mu_{nf}\frac{\partial^2 u}{\partial y^2} + \frac{16\sigma^* T_\infty^3}{3k^*}\frac{\partial^2 T}{\partial y^2} + \sigma_{nf}B_0^2 u^2 + \frac{\mu_{nf}}{k_{fh}}u^2 + \frac{Fs}{k_{fh}}u^3, \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - K_r(C - C_\infty). \tag{4}$$

Boundary conditions associated with the above PDEs are [27]:

$$\left\{\begin{array}{c} y = 0: \quad u = U_w = cx, \quad v = v_w, \quad T = T_w, \quad C = C_w \\ y \to \infty: u = 0, \quad v = 0, \quad T = T_\infty, \quad C = C_\infty \end{array}\right\}, \tag{5}$$

where $v_w > 0$ and $v_w < 0$ represents injection and suction at the surface respectively. Now, using the following similarity transformations:

$$\left\{\begin{array}{c} \eta = \sqrt{\frac{c}{\nu_{bf}}}y, \quad u = cxf'(\eta), \quad v = -\sqrt{c\nu_{bf}}f(\eta), \\ T = T_\infty + \theta(\eta)(T_w - T_\infty), \quad C = C_\infty + \phi(\eta)(C_w - C_\infty) \end{array}\right\}, \tag{6}$$

Equations (2)–(5) are transformed to the following non-dimensional form:

$$f''' - wf' - \lambda_1 M^2 f' - \lambda_2[(Fr + 1)(f')^2 - ff''] = 0, \tag{7}$$

$$(\lambda_6 + Rd)\theta'' + PrEc[\lambda_4 f''2 + (w\lambda_4 + \lambda_7 M^2)(f')^2 + FrKf'3] + \lambda_5 Prf\theta' = 0, \tag{8}$$

$$\phi'' + Scf\phi' - ScG\phi = 0. \tag{9}$$

the non-dimensional boundary conditions are

$$\left\{\begin{array}{c} f(0) = Fw, \quad f'(0) = 1, \quad \theta(0) = 1, \quad \phi(0) = 1, \\ f'(\infty) = 0, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0 \end{array}\right\}, \tag{10}$$

where

$$\left\{\begin{array}{c} w=\frac{v_{bf}}{ck_{fh}},\quad Fr=C_bk_{fh}^{-1/2}=F.x,\quad M^2=\frac{\sigma_{bf}B_0^2}{\rho_{bf}c},\quad Sc=\frac{\nu_{bf}}{D},\\ Rd=\frac{16\sigma^*T_\infty^3}{3k^*\kappa_{bf}},\quad G=\frac{K_r}{c},\quad Pr=\frac{\mu_{bf}(C_p)_{bf}}{\kappa_{bf}},\quad Ec=\frac{U_w^2}{(C_p)_{bf}(T_w-T_\infty)},\\ K=\frac{1}{k_{fh}},\lambda_1=\frac{\sigma_{nf}}{\sigma_{bf}}.\frac{\mu_{bf}}{\mu_{nf}},\lambda_2=\frac{\nu_{bf}}{\nu_{nf}},\quad \lambda_3=\frac{(\rho\beta)_{nf}}{(\rho\beta)_{bf}}.\frac{\mu_{bf}}{\mu_{nf}},\quad \lambda_4=\frac{\mu_{nf}}{\mu_{bf}},\\ \lambda_5=\frac{(\rho C_p)_{nf}}{(\rho C_p)_{bf}},\quad \lambda_6=\frac{\kappa_{nf}}{\kappa_{bf}},\quad \lambda_7=\frac{\sigma_{nf}}{\sigma_{bf}},\quad Fw=-\frac{v_w}{\sqrt{cv_f}} \end{array}\right\}. \tag{11}$$

2.1 Engineering Interest Quantities

Skin friction, Nusselt number and Sherwood number at the surface are given by

$$\left\{C_{fx}=\frac{\tau_w}{\rho_{bf}U_w^2},\ Nu_x=\frac{xq_w}{\kappa_{bf}\left(T_w-T_\infty\right)},\ Sh_x=-\frac{xq_m}{D\left(C_w-C_\infty\right)}\right\}, \tag{12}$$

where τ_w, q_w and q_m are shear stress, heat flux and mass flux respectively,

$$\left\{\tau_w=\mu_{nf}\frac{\partial u}{\partial y},\ q_w=-\kappa_{nf}\frac{\partial T}{\partial y}-\frac{16\sigma^*T_\infty^3}{3k^*\kappa_{bf}}\frac{\partial T}{\partial y},\ q_m=-D\left(\frac{\partial C}{\partial y}\right)_{y=0}\right\}. \tag{13}$$

Using (6) and (13) we get:

$$\left\{C_{fx}Re_x^{\frac{1}{2}}=\lambda_4 f''(0),\ Nu_xRe_x^{-\frac{1}{2}}=-\left[\lambda_6+Rd\right]\theta'(0),\ Sh_xRe_x^{-\frac{1}{2}}=\phi'(0)\right\} \tag{14}$$

3 Research Methodology

The non-linear system of ODEs given by (7)–(9) are solved numerically using BVP4C solver in MATLAB. BVP4C is a MATLAB program that uses the finite difference code which implements the three stage Lobatto Illa formula. This method provides a solution of fourth order which is accurate and uniform in the range of integration. The problem is solved by taking $\Delta\eta = 0.001$ and the results of convergency are maintained within the tolerance limit of 10^{-6}. The obtained result satisfies the asymptotic boundary conditions that accentuates the accuracy of the numerical computations. The transformed problem along with boundary conditions is given below
Let,

$$\left\{\begin{array}{c} f=y_1,\quad f'=y_2,\quad f''=y_3\\ \theta=y_4,\quad \theta'=y_5,\quad \phi=y_6,\quad \phi'=y_7 \end{array}\right\}. \tag{15}$$

From the system of O.D.E (7)–(9) with the above considerations, we get

$$y_3' = wy_2 + \lambda_1 M^2 y_2 + \lambda_2 \left[(Fr + 1)\, y_2^2 - y_1 y_3 \right], \tag{16}$$

$$y_4' = \frac{1}{\lambda_6 + Rd} \left(-PrEc\lambda_4 y_3^2 + w\lambda_4 + \lambda_7 M^2 y_2^2 - \lambda_5 Pr y_1 y_5 - FrEcPr y_2^3 K \right), \tag{17}$$

$$y_6' = -Scy_1 y_7 + ScGy_6. \tag{18}$$

With the boundary conditions:

$$\left\{ \begin{array}{c} y_1(0) = Fw, \quad y_2(0) = 1, \quad y_4(0) = 1, \quad y_6(0) = 1 \\ y_2(\infty) = y_4(\infty) = 0, \quad y_7(\infty) = 0 \end{array} \right\}. \tag{19}$$

3.1 Stability Analysis and Validation

The stability and convergence of the adopted finite difference scheme has been presented in Table 3 for different step sizes and it is evident that the solution is stable and convergent since the boundary conditions are satisfied by the velocity, temperature and concentration. Tables 4 and 5 cite the correctness of the current flow model by comparing the numerical results obtained with the results obtained by Shaiq et al. [31], Manjunatha et al. [32] and Chakrabarti and Gupta [33]. The behaviour of the Nusselt number for different Pr for the present work is comparable with earlier studies.

Table 3 Stability Test

$h = 0.001$				$h = 0.0001$			
η	f'	θ	ϕ	η	f'	θ	ϕ
0	1.0000	1.0000	1.0000	0	1.0000	1.0000	1.0000
0.0490	0.9433	0.8981	0.8580	0.4999	0.5410	0.2365	0.1948
0.5990	0.4751	0.1654	0.1381	0.8999	0.3130	0.0494	0.0472
1.1990	0.1943	0.0130	0.0156	1.4999	0.1050	0.0031	0.0055
1.8990	0.0185	0.0002	0.0004	1.7999	0.0377	0.6470	0.0010
1.9090	0.0161	0.0001	0.0004	1.9009	0.0170	0.0002	0.0003
1.9990	0.0001	0.0000	0.0000	1.9499	0.0081	0.0001	0.0002
2.0000	0	0	0.0000	2.0000	0	0	0.0000

4 Result and Discussions

In this section, the impact of various parameters on the mobility and heat exchange of *SWCNT–EG* nanofluid are covered. Tables 6 and 7 illustrate the numerical values of skin friction and Sherwood number by varying different parameters. Higher values of skin friction have been observed for higher values of w, Fr, and M, whereas Rd lowers the skin friction. In Table 7, the impact of the Schmidt number Sc and the chemical reaction parameter G on the Sherwood number is shown. It is observed that when G and Sc values improve, the Sherwood number also falls.

The Influence of M and φ over $f'(\eta), \theta(\eta)$ and $\varphi(\eta)$ has been presented in Fig. 2a–c. Due to the application of the Lorentz force, a decline in the velocity profile has been detected, while a rise in the temperature as well as the concentration profile has been observed. When a magnetic field is applied to an electrically conducting fluid, it creates a drag-like force known as the Lorentz force, which opposes the transport phenomenon. This force reduces nanofluid velocity within the boundary layer and enhances the temperature of the fluid. The impact of volume fraction over the temperature profile has been suppressed by the magnetic parameter.

The impact of Fr and Fw over $f'(\eta), \theta\ (\eta)$ and $\varphi(\eta)$ has been demonstrated in Fig. 3a–c. The application of Forchheimer drag creates resistance to the fluid flow, thus slowing their movement. Higher Fr has a greater impact on the impedance to velocity, which reduces fluid velocity. An increase in the temperature profile for higher Fr is due to the resistance offered by the drag force, which generates thermal energy for the molecules of the fluid, resulting in a higher temperature. A Slight increment in the concentration profile has also been observed for augmented values of Fr. The Application of wall suction reduces the velocity of the nanofluid. It is also evident that with the application of suction, the temperature and concentration of the nanofluid increase as compared to blowing.

The influence of various parameters over $C_{fx}Re_x^{1/2}$, $Nu_xRe_x^{-1/2}$ and $Sh_xRe_x^{-1/2}$ has been demonstrated in Fig. 4a–c. The Addition of $SWCNT$ nanoparticles to the base fluid increased the rate of heat transfer from the surface, whereas $C_{fx}Re_x^{1/2}$ has been declined with the addition of nanoparticles. The Magnetic parameter plays an important role in the velocity and concentration gradient at the surface. As the magnetic field strength increases, the skin friction decreases significantly, whereas the concentration gradient at the surface increases. This is technically possible due to the magnetic field's ability to slow the velocity field and lessen the force of fric-

Table 4 Comparison table for Nusselt number

Pr	Manjunatha et al. [32]	Shaiq et al. [31]	Present
0.7	–	0.4539	0.4716
2	0.9113	0.9114	0.9107
6.13	1.7597	–	1.7592
7	1.8954	1.8954	1.8949

Table 5 Comparison table for Nusselt number for $Pr = 0.73$, $Fw = 1$ and rest parameters are zero

M	Chakrabarti and Gupta [33]	Present
5	0.9154	0.9225
10	0.8837	0.8927
15	0.8645	0.8752
20	0.8522	0.8633

tion at the wall. Rd has been found to significantly affect the Nusselt number in the presence of nanoparticles. This characteristic is explained by the fact that Rd enhances the Nusselt number, and nanoparticles in the base fluid increase the thermal conductivity of the base fluid. With the application of suction at the surface, the rate of heat transfer increases. The influence of Sc and G over $Sh_x Re_x^{-1/2}$ is opposite to each other. The Schmidt number is a dimensional number that given by $Sc = \frac{Momentum\ \ diffusivity}{Mass\ \ diffusivity}$. Since Sc is inversely related to mass diffusivity. Thus, rise in Sc decreases the concentration gradient at the surface. In an endothermic reaction, the mass flux to the molecules of the species is diffused from the fluid region to the surface, reducing the molar concentration, and hence, the Sherwood number has declined for higher values of G.

Table 6 Outcomes of $|C_{fx} Re_x^{1/2}|$ when $Pr = 7$, $Gr = 1$, $\varphi_{SWCNT} = 0.04$, $K = 10^4$, $Sc = 0$, $G = 0$

$\lvert C_{fx} Re_x^{1/2} \rvert$						
w	Fr	M	Rd	$Fw = 1.5$	$Fw = 0$	$Fw = -1.5$
0.1	0.5	2.0	0.2	3.2056	2.1872	1.4055
0.5				3.3176	2.3058	1.5216
1.1				3.4251	2.4194	1.6329
0.1	0.2			3.2563	2.2352	1.4461
	1.0			3.3600	2.3593	1.5836
	1.8			3.4604	2.4777	1.7127
	0.5	0.0		2.0623	0.9834	0.2799
		0.5		2.1787	1.0861	0.3678
		1.0		2.4368	1.3611	0.6132
		1.5		2.7893	1.7440	0.9747
			0.0	3.2175	2.1968	1.4050
			0.5	3.1897	2.1750	1.4062
			1.0	3.1653	2.1589	1.4072

Table 7 Outcomes of $Sh_x Re_x^{-1/2}$ when $Pr = 7$, $Gr = 1$, $\varphi_{SWCNT} = 0.04$, $K = 10^4$, $Sc = 0$, $G = 0$

$Sh_x Re_x^{-1/2}$				
Sc	G	$Fw = 0.6$	$Fw = 0$	$Fw = -0.6$
0.2	0.2	–0.2107	–0.2105	–0.1830
0.6		–0.3810	–0.3790	–0.2335
1		–0.5715	–0.5680	–0.2694
0.2	0.6	–0.3090	–0.2791	–0.2513
	1	–0.7525	–0.7221	–0.6948
	1.4	–0.4445	–0.4141	–0.3861

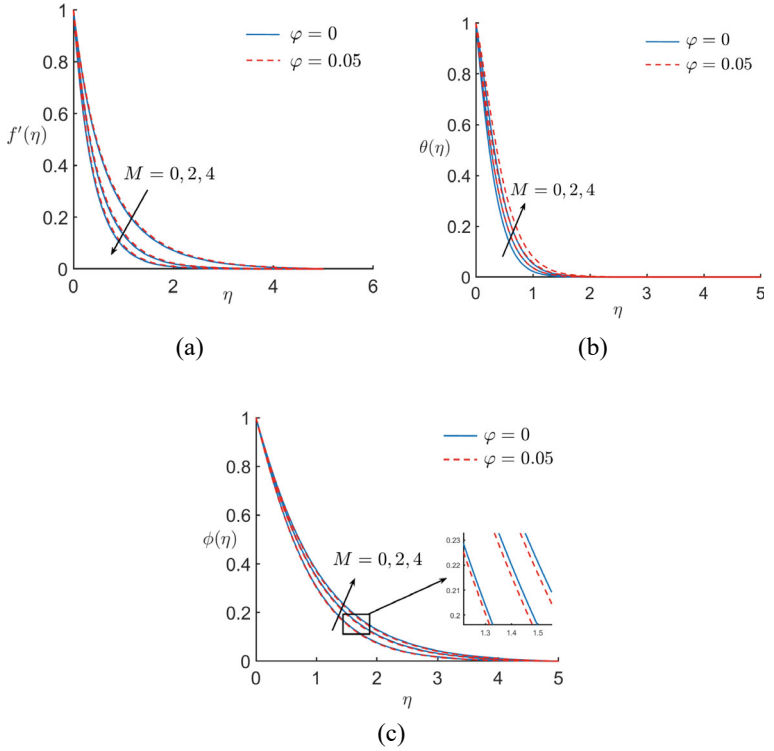

Fig. 2 **a–c** Influence of M and φ over $f'(\eta)$,$\theta(\eta)$ and $\phi(\eta)$

5 Conclusion

The current model provides thorough numerical investigations of the heat transfer and Darcy–Forchheimer flow along a stretchable surface of pores that is affected by thermal radiation, suction/blowing and magnetic fields with first-order chemical

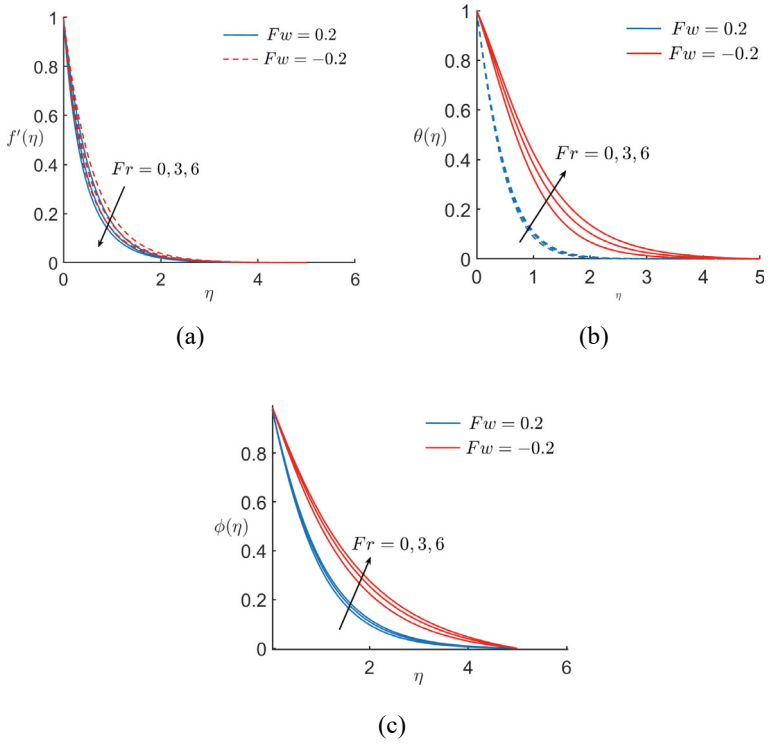

Fig. 3 **a–c** Influence of Fr and Fw over $f'(\eta)$,$\theta(\eta)$ and $\phi(\eta)$

reactions under endothermic and exothermic conditions. For the model's formulation, a system of Cartesian coordinates is taken into account. The system of partial differential equations has been transformed into ODEs by using appropriate similarity transformations, and finally, the system of ODEs has been solved via the MATLAB bvp4c routine. The following are the primary findings of the current study:

(1) With the augmented values of M, velocity of the nanofluid decelerates, while temperature and concentration profile increases.

(2) Fr aids in enhancing temperature and concentration profile, but it slows down the velocity of the fluid.

(3) Augmented values of M, Fw and φ reduces skin friction at the surface.

(4) Nusselt number rises with the augmented values of Rd, φ and Fw.

(5) Sherwood number decays for escalating values of G and Sc, while M rises the Sherwood number.

In future, the current investigation can be extended to

(1) study the impact of new physical factors on the flow behaviour of the nanofluid,

(2) study the effect of the size and shape of the nanoparticles on the physical attributes,

(3) study the phenomenon of Non-Newtonian fluid model with various aspects.

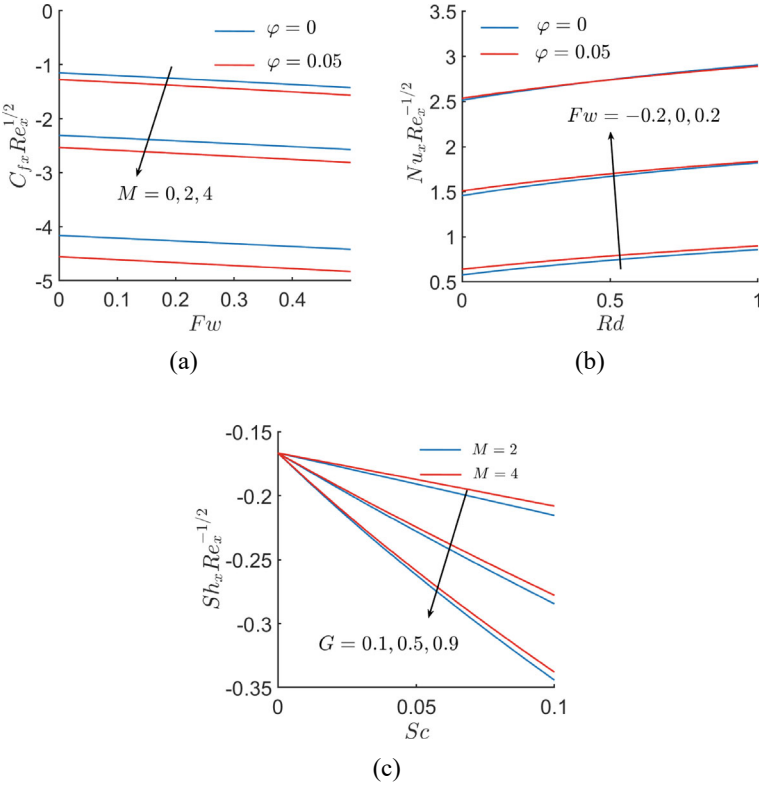

Fig. 4 Influence of **a** M, φ, Fw over $C_{fx}Re_x^{1/2}$ **b** M, φ, Fw over $Nu_xRe_x^{-1/2}$ **c** M, G and Sc over $Sh_xRe_x^{-1/2}$

In engineering areas, the Nusselt number plays an important role in studying the heat transmission from equipments. As a result of the current investigation, *SWCNT* and suction parameter have elevated the Nusselt number, which can be an important factor in enhancing the heat transfer rate. This study has significant applications in the areas of medical and engineering sciences. *SWCNT* are used for cancer treatment, so by considering blood as a base fluid, one can also explore its applications in medical science. Due to the higher thermal conductivity of *SWCNT* it becomes a good candidate to be used in heat exchanger devices.

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Big Rip in Tsallis Holographic Dark Energy Model Within the Framework of General Relativity

Chandra Rekha Mahanta and Manash Pratim Das

Abstract In this chapter, utilizing the general relativity framework, we investigate cosmological models for both non-interacting and interacting Tsallis holographic dark energy (THDE) and pressureless cold dark matter in LRS Bianchi type II space-time. In THDE density, we consider the Hubble horizon as the IR cut-off. Complete solutions of Einstein's filed equations are achieved for some particular cases corresponding to the value of an integrating constant C. For $C = 0$, we get decelerating universe throughout the evolution. While for $C \neq 0$, we find that the present universe is passing through an accelerating phase of expansion. However the expansion rate of the universe will decelerate at late times. For both the interacting and non-interacting case, we find that the universe contains phantom energy in the early universe and non-phantom energy at present and late times. Also for both interacting and non-interacting cases, the coincidence problem can be resolved.

1 Introduction

A number of astrophysical observations, including Supernovae Type Ia (SN Ia) [1–3], Cosmic Microwave Background (CMB) [4, 5], Large Scale Structure (LSS) [6, 7], observational data obtained from Planck Collaboration Results 2015 [8] etc. sturdily suggest us about the late time accelerated expansion of the universe. In accordance with general relativity, if the cosmos is filled with only matter and radiation, the expansion of the universe should have a small acceleration [9, 10]. In order to explain the observed accelerated phase of the universe's expansion, a further component with a significant negative pressure is required. In the framework of standard cosmology,

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the universe is supposed to be dominated by a strange fluid which exerts negative pressure dubbed dark energy (DE).

Among the dark energy candidates proposed in literature, the cosmological constant Λ plays an important role in explaining the late time cosmic expansion, but due to the fine tuning as well as the cosmic coincidence problem, some other candidates of dark energy have to be brought under consideration [11–13]. Several researchers have introduced a number of dynamically evolving scalar field models such as quintessence [14], k-essence [15], phantom [16], Tachyon [17], dilatonic ghost condensate model [18] etc. in quest of dark energy.

Another significant candidate that emerges from the Holographic Principle and proposes that the limit imposed by the formation of a black hole causes the short distance (UV) cut-off to be related to the long distance infrared (IR) cut-off in an effort to resolve the dark energy problem has been put forth in literature. Li in 2004 investigated a dark energy model based on the Holographic Principle known as "Holographic dark energy principle" [19]. The energy density of holographic dark energy (HDE) is characterized as $\rho_{HDE} = 3c^2 M_p^2 L^{-2}$, where L is the IR cut-off, M_p^2 is the Planck mass and c is a numerical constant. Tsallis and Cirto in 2013 proposed a new model of holographic dark energy called Tsallis Holographic dark energy (THDE) model by using Tsallis generalized entropy, $S_\delta = \gamma A^\delta$, where γ is an unknown constant and δ is a non-additive parameter [20]. Using Tsallis entropy in Holographic Principle, the energy density of the Tsallis holographic dark energy can be obtained as $\rho_{THDE} = DL^{2\delta-4}$, where D is an unknown parameter [21, 22].

In recent times, Korunur [23] has studied Tsallis holographic dark energy in Bianchi type III space-time considering the deceleration parameter, the transition of which from deceleration to acceleration phase happens at $z = 0.256$. Mamon et al. [24] studied a theoretical model for interacting Tsallis holographic dark energy (THDE) whose infrared cut-off scale is set by the Hubble length. Bhattacharjee [25] investigated the dynamics of Tsallis holographic dark energy (THDE) and Rényi holographic dark energy (RHDE) for a scale factor evolving with a composite power law-exponential (hybrid) form, as dictated by a non-linear interaction in the FRW space-time. Dubey et al. [26] investigated Tsallis holographic dark energy (THDE), infrared cut-off taking the Hubble horizon with a hybrid expansion law in the Bianchi-I (axially symmetric) anisotropic model. Ayman [27] studied Tsallis Holographic Dark Energy with Granda-Oliveros Scale in $(n + 1)$ Dimensional FRW Universe. Several other researchers have studied Tsallis holographic dark energy in different contexts to study the evolution of the universe, late time cosmic expansion and the related aspects.

Though the FLRW model is the best one for describing the large-scale structure of the universe and the current universe appears to be homogenous and isotropic, there is no conformed observable data that rules out the possibility of an anisotropic universe. The measurements of Cosmic Microwave Background (CMB) anisotropy suggest the existence of anisotropies in the early universe. The concern of global anisotropy can also be settled by incorporating anisotropy to the homogeneous and isotropic flat Friedmann-Lemaître-Robertson-Walker (FLRW) model in a suitable manner. Since

Bianchi type cosmological models are the simplest models with anisotropic background and these models play a significant role in the description of large scale behaviour of the universe, therefore, in an effort to figure out the dynamics of the evolving universe, spatially homogeneous and anisotropic Bianchi models are usually considered. Bianchi type models can completely explain the effect of anisotropy and also present a middle way between the homogeneous and isotropic FLRW models and inhomogeneous and anisotropic models. Hence, spatially homogeneous and anisotropic Bianchi cosmological models have been widely studied by many authors in different contexts.

In this paper, we focus our attention to study the spatially homogeneous and anisotropic Bianchi type II universe filled with both interacting and non-interacting Tsallis holographic dark energy (THDE) and cold dark matter. In our study the evolution of the universe is investigated by taking the Tsallis holographic dark energy density in the form $\rho_{THDE} = DL^{2\delta-4}$, where D is an unknown parameter.

We organize the paper in the following manner: In Sect. 2, against the background of LRS Bianchi type II uinverse, we derive the cosmic evolution equations from the Einstein field equations. We solve the field equations in Sect. 3 by imposing a suitable condition relating the directional scale factors. In Sect. 4, we provide solutions of the field equations for two special cases. Two sub cases for non-interacting and interacting THDE with cold dark matter are studied, and the physical as well as geometrical characteristics of the models are thoroughly addressed. Finally, we conclude the paper in Sect. 5 with a brief discussion.

2 Metric and Field Equations

We consider the LRS Bianchi type II space-time described by the line-element

$$ds^2 = -dt^2 + A^2(dx^2 - zdy)^2 + B^2(dy^2 + dz^2) \tag{1}$$

where A and B are the directional scale factors and are functions of the cosmic time t alone.

We assume that the universe is filled with a mixture of pressureless cold dark matter and Tsallis holographic dark energy of density ρ_{THDE} with the identification of IR cut-off as Hubble horizon given by

$$\rho_{THDE} = DH^{-2\delta+4} \tag{2}$$

where D is an unknown parameter.

In natural units ($8\pi G = 1, c = 1$), the Einstein field equations are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} + \overline{T}_{ij} \tag{3}$$

where R_{ij} is the Ricci tensor, R, the Ricci scalar curvature, and T_{ij} and $\overline{T}_{ij}$ are the energy-momentum tensors of pressureless cold dark matter and Tsallis holographic dark energy respectively. The energy-momentum tensors T_{ij} and $\overline{T}_{ij}$ are given by

$$T_{ij} = \rho_m u_i u_j \tag{4}$$

$$\overline{T}_{ij} = (\rho_{THDE} + p_{THDE}) + g_{ij} p_{THDE} \tag{5}$$

where ρ_m and ρ_{THDE} are the energy densities of cold dark matter and Tsallis holographic dark energy (THDE) respectively and p_{THDE} is the pressure of THDE.

In comoving coordinates, the Einstein field equations (3) for the metric (1) are obtained as

$$2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + (\frac{\dot{B}}{B})^2 - \frac{1}{4}\frac{A^2}{B^4} = \rho_m + \rho_{THDE} \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{1}{4}\frac{A^2}{B^4} = -p_{THDE} \tag{7}$$

$$2\frac{\ddot{B}}{B} + (\frac{\dot{B}}{B})^2 - \frac{3}{4}\frac{A^2}{B^4} = -p_{THDE} \tag{8}$$

where an over dot indicates differentiation with respect to cosmic time t.

The vanishing of the divergence of Einstein tensor yields

$$T^i_{j,i} + \overline{T}^i_{j,i} = 0 \tag{9}$$

which gives the continuity equation

$$\dot{\rho}_m + \dot{\rho}_{THDE} + 3H(\rho_m + \rho_{THDE} + p_{THDE}) = 0 \tag{10}$$

where H is the average Hubble parameter, $H_1 = \frac{\dot{A}}{A}$, $H_2 = H_3 = \frac{\dot{B}}{B}$ being the directional Hubble parameter.

For the metric (1), the average Hubble parameter H, the expansion scalar θ, the mean anisotropy parameter A_m, the shear scalar σ, and the deceleration parameter q are defined by

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) \tag{11}$$

$$\theta = 3H = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \tag{12}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\frac{(H_i - H)^2}{H^2} = \frac{2}{9H^2}(\frac{\dot{A}}{A} - \frac{\dot{B}}{B})^2 \tag{13}$$

$$\sigma^2 = \frac{1}{2}\left[\sum_{i=1}^{3} H_i^2 - \frac{1}{3}\theta^2\right] = \frac{3}{2}A_m H^2 \tag{14}$$

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 \tag{15}$$

3 Solution of the Field Equations

We have four Eqs. (2) and (6)–(8) and five unknowns A, B, ρ_m, ρ_{THDE} and p_{THDE}. In order to obtain a complete solution of the field equations, one more physical condition relating the unknown parameters is required. For this purpose, we relate the directional scale factors as

$$A = lB^m \tag{16}$$

for some positive constants l and m.

Then from Eqs. (7) and (8), we have

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{A^2}{B^4} = 0 \tag{17}$$

Defining a parameter V by

$$lV = AB^2 \tag{18}$$

where l is a constant and using (16), we get

$$A = lV^{\frac{m}{m+2}}, B = V^{\frac{1}{m+2}} \tag{19}$$

Using (19) in Eq. (17), we get

$$\ddot{V} = \frac{l^2(m+2)}{1-m}V^{\frac{3m-2}{m+2}} \tag{20}$$

From which we obtain

$$\dot{V}^2 = \frac{l^2(m+2)^2}{2\,m(1-m)}[V^{\frac{4m}{m+2}} + C] \tag{21}$$

where C is a constant of integration. Equation (21) yields

$$\dot{V} = \frac{l(m+2)}{\sqrt{2\,m(1-m)}}\sqrt{V^{\frac{4m}{m+2}} + C} \tag{22}$$

This shows that $m < 1$ and therefore, $0 < m < 1$. Integrating Eq. (22) and suppressing the integrating constant, we get

$$\int \frac{dV}{\sqrt{V^{\frac{4m}{m+2}} + C}} = \frac{l(m+2)t}{\sqrt{2\,m(1-m)}} \tag{23}$$

Now, to evaluate the integral on the left hand side of the Eq. (23), we consider two special cases: $C = 0$ and $C \neq 0$.

Case I: When $C = 0$, the Eq. (23) reduces to

$$\int V^{\frac{-2m}{m+2}} dV = \frac{l(m+2)t}{\sqrt{2\,m(1-m)}} \tag{24}$$

On integration, we get

$$V = kt^{\frac{2+m}{2-m}} \tag{25}$$

where $k = [\frac{l(m+2)}{\sqrt{2m(1-m)}}]^{\frac{2+m}{2-m}}$ is a constant.

Using (25), from Eq. (19), we get

$$A = lk^{\frac{m}{2+m}} t^{\frac{m}{2-m}} \tag{26}$$

$$B = k^{\frac{1}{2+m}} t^{\frac{1}{2-m}} \tag{27}$$

Thus the cosmological parameters for $C = 0$ are obtained as

$$H = \frac{2+m}{3(2-m)t} \tag{28}$$

$$\theta = \frac{2+m}{(2-m)t} \tag{29}$$

$$A_m = \frac{2(1-m)^2}{(2+m)^2} \tag{30}$$

$$\sigma^2 = \frac{(1-m)^2}{3(2-m)^2 t^2} \tag{31}$$

$$\rho_{THDE} = D\left[\frac{2+m}{3(2-m)t}\right]^{-2\delta+4} \tag{32}$$

$$\rho_m = \frac{(1+m)(2+m)}{2(2-m)^2t^2} - D\left[\frac{2+m}{3(2-m)t}\right]^{-2\delta+4} \tag{33}$$

$$q = \frac{4(1-m)}{m+2} \tag{34}$$

But $q > 0$ for $0 < m < 1$, so this model does not lead to accelerated phase of expansion of the universe for any t.

Case II: When $C \neq 0$, the left side of the Eq. (23) can be evaluated easily if $\frac{4m}{m+2} = 0, 1, 2$ or $m = 0, \frac{2}{3}, 2$. But, since $0 < m < 1$, so, we take $m = \frac{2}{3}$ in particular, to obtain exact solutions of the field equations. Then integration of the Eq. (23) yields

$$V = 4l^2t^2 - C \tag{35}$$

Therefore, the directional scale factors are given by

$$A = l[4l^2t^2 - C]^{\frac{1}{4}} \tag{36}$$

$$B = [4l^2t^2 - C]^{\frac{3}{8}} \tag{37}$$

And the cosmological parameters for $C \neq 0$ are obtained as

$$H = \frac{2t}{3(t^2 - \frac{C}{4l^2})} \tag{38}$$

$$\theta = \frac{2t}{t^2 - \frac{C}{4l^2}} \tag{39}$$

$$A_m = \frac{1}{32} \tag{40}$$

$$\sigma^2 = \frac{t^2}{(t^2 - \frac{C}{4l^2})^2} \tag{41}$$

$$\rho_{THDE} = D(\frac{2t}{3(t^2 - \frac{C}{4l^2})})^{-2\delta+4} \tag{42}$$

$$q = \frac{1}{2} + \frac{3}{8}\frac{C}{l^2t^2} \tag{43}$$

For accelerating expansion of the universe, we must have $q < 0$ and therefore from (43), $C < 0$. Also, $q > 0$ for $t < -\frac{\sqrt{-3C}}{2l}$ and $t > \frac{\sqrt{-3C}}{2l}$, and $q < 0$ for $-\frac{\sqrt{-3C}}{2l} < t < \frac{\sqrt{-3C}}{2l}$. H, θ, σ are maximum and V is minimum at $t = \frac{\sqrt{-C}}{2l}$. $H = 0$ at $t = 0$,

which is not a physically viable condition. Therefore, by taking $-C = K$, where $K > 0$, we find that the universe starts at $t = \frac{\sqrt{K}}{2l}$ with volume $V = 2K$ and expands with acceleration till $t < \frac{\sqrt{3K}}{2l}$ and then expands with deceleration till the end.

From the observed data it is clear that currently our universe is expanding with acceleration, but we cannot deny about the deceleration in future. The universe is full of mystery and the true nature of dark energy is yet unknown. The universe has already transitioned from deceleration to acceleration and in near future it may have to pass from acceleration to deceleration. Our model describes a universe which passed through phase of accelerated expansion for a certain period of time as well as at current time and subsequently transitions from accelerating expansion to decelerating expansion. So, this model will be suitable to describe such type of universe.

From Fig. 2, it is clear that Tsallis HDE is maximum at the start of the universe and it decreases as time evolves and asymptotically approaches towards 0 at late time.

Sub case II(a): Non-interacting Tsallis HDE

In this case, from the energy conservation equation, we get

$$\dot{\rho}_m + 3H\rho_m = 0 \tag{44}$$

$$\dot{\rho}_{THDE} + 3H(\rho_{THDE} + p_{THDE}) = 0 \tag{45}$$

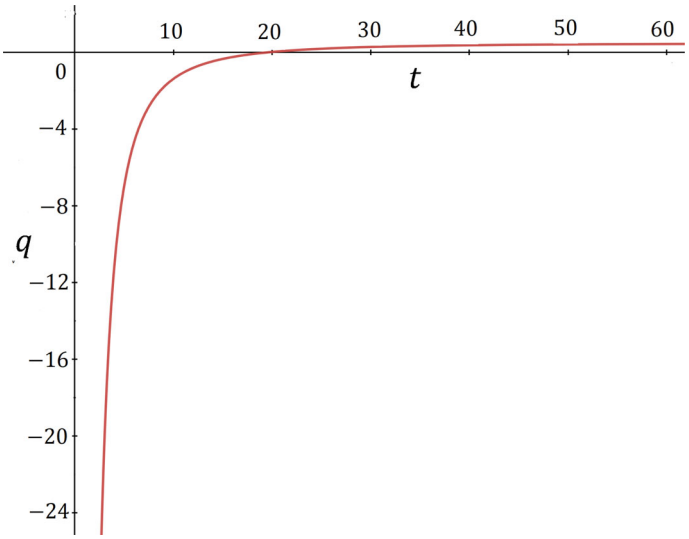

Fig. 1 Deceleration parameter versus cosmic time t graph with $K = 500, l = 1$

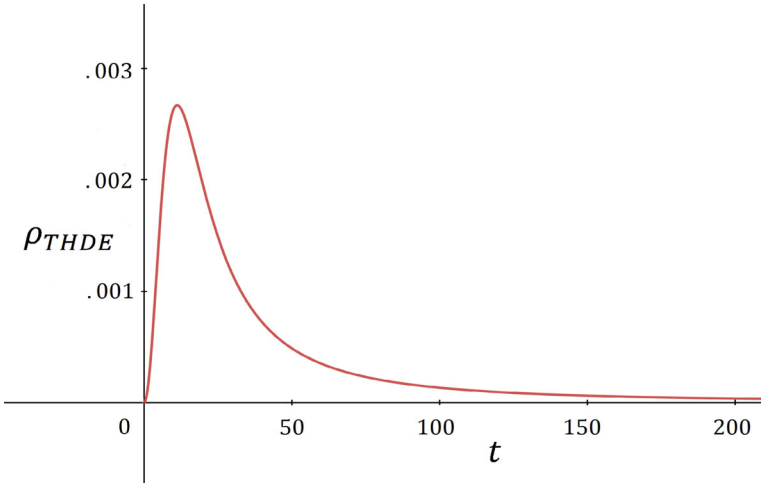

Fig. 2 ρ_{THDE} versus cosmic time t graph with $D = 3$, $K = 1, l = 1, \delta = 1$

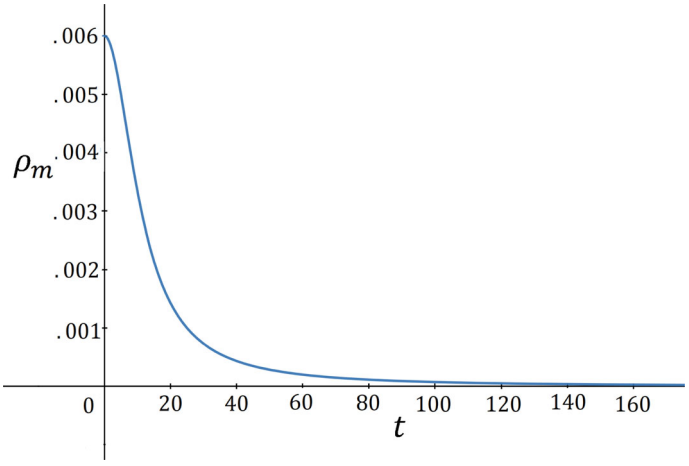

Fig. 3 The plot of ρ_m versus cosmic time t with $c_1 = 3, l = 1, K = 500$

From Eq. (44), we get

$$\rho_m = \frac{c_1}{4l^3(t^2 + \frac{K}{4l^2})} \tag{46}$$

where $c_1 > 0$ is an integrating constant.

From Fig. 3 it is clear that ρ_m decreases as time evolves and tends to 0 at late times.

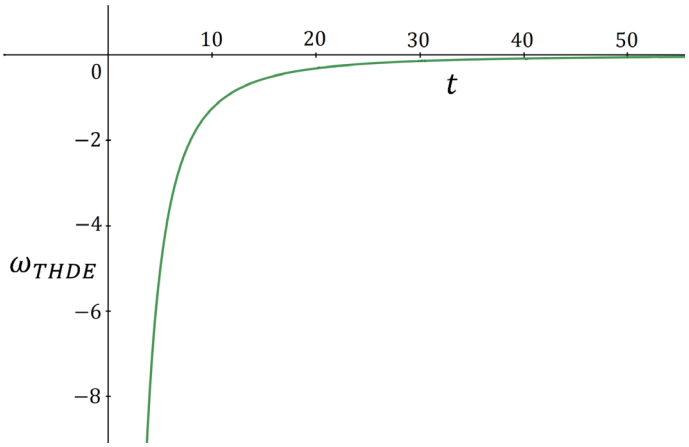

Fig. 4 The plot of ω_{THDE} versus cosmic time t with $l = 1$, $K = 500$, $\delta = 1$

From Eq. (45), we get the EoS parameter of THDE as

$$\omega_{THDE} = \frac{p_{THDE}}{\rho_{THDE}} = -1 - \frac{1}{3H}\frac{\dot{\rho}_{THDE}}{\rho_{THDE}} = -1 + (-2\delta + 4)\left(1 - \frac{t^2 + \frac{K}{4l^2}}{2t^2}\right) \tag{47}$$

From Fig. 4 it is clear that the universe contains phantom energy in the early universe and non-phantom energy at present as well as at late time. Also, for $-1 < \omega_{THDE} < -\frac{1}{3}$, we have

$$\frac{\sqrt{K}}{2l} < t < \frac{\sqrt{3K}}{2l} \tag{48}$$

Hence, during this period the THDE behaves like quintessence.

The energy density parameters Ω_m and Ω_{THDE} are given by

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{3c_1}{16l^3}[1 + \frac{K}{4l^2t^2}] \tag{49}$$

$$\Omega_{THDE} = \frac{\rho_{THDE}}{3H^2} = \frac{3D(t^2 + \frac{K}{4l^2})^2}{4t^2}[\frac{2t}{3(t^2 + \frac{K}{4l^2})}]^{-2\delta+4} \tag{50}$$

Hence, the total energy density is obtained as

$$\Omega = \frac{3c_1}{16l^3}\left[1 + \frac{K}{4l^2t^2}\right] + \frac{3D(t^2 + \frac{K}{4l^2})^2}{4t^2}\left[\frac{2t}{3(t^2 + \frac{K}{4l^2})}\right]^{-2\delta+4} \tag{51}$$

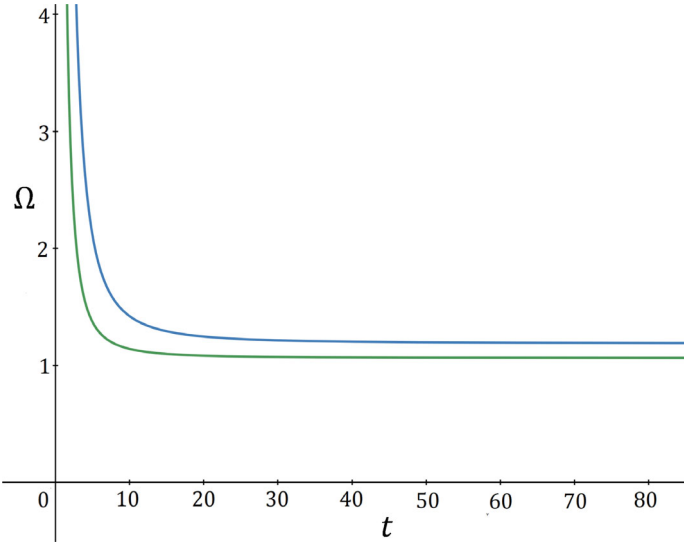

Fig. 5 The plot of Ω versus cosmic time t with $l = 1$, $K = 500$, $\delta = 1$, $D = 3$, $c_1 = 1, \frac{1}{3}$

For relatively smaller values of the integrating constant c_1, the total energy density tends to 1 at late time, i.e., the late time universe is flat. In Fig. 5, the green line correspondence to $c_1 = \frac{1}{3}$ and blue line corresponds to $c_1 = 1$.

The cosmic coincidence parameter is given by

$$r = \frac{\rho_m}{\rho_{THDE}} = \frac{\frac{c_1}{4l^3(t^2+\frac{K}{4l^2})}}{D\left[\frac{2t}{3(t^2+\frac{K}{4l^2})}\right]^{-2\delta+4}} \tag{52}$$

Equation (52) and the Fig. 6 show that at $t \to \infty$, $r = a$ constant for $1 \leq \delta < 2$. For $\delta < 1$, r is divergent. And for $\delta = 2$, $r \to 0$. Hence, the cosmic coincidence problem can be resolved at late time for $1 \leq \delta < 2$.

Sub case II(b): Interacting Tsallis HDE

For interacting THDE and cold dark matter, we take

$$\dot{\rho}_m + 3H\rho_m = Q \tag{53}$$

Therefore, from (10), we get

$$\dot{\rho}_{THDE} + 3H(\rho_{THDE} + p_{THDE}) = -Q \tag{54}$$

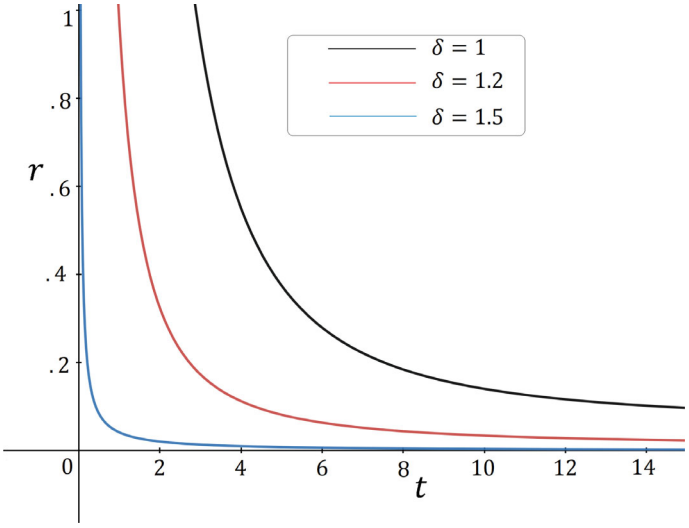

Fig. 6 The plot of coincidence parameter r versus cosmic time t with $l = 1$, $K = 500$, $D = 3$, $c_1 = \frac{1}{3}$

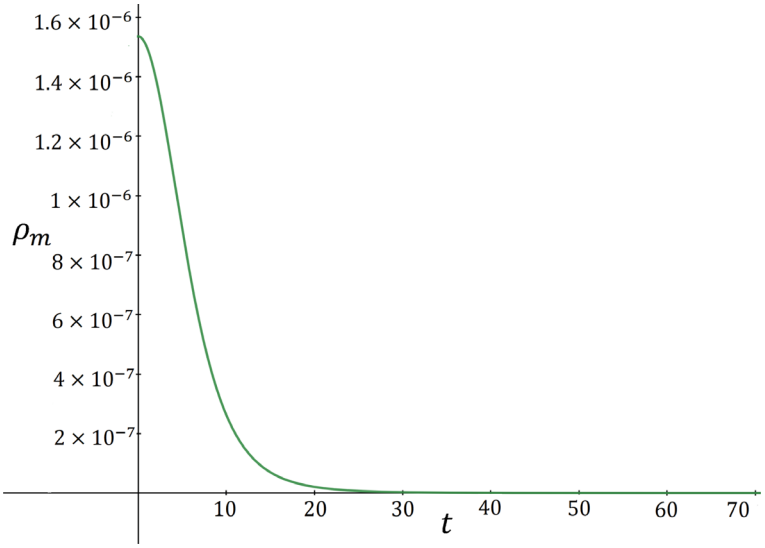

Fig. 7 The plot of coincidence parameter r versus cosmic time t with $l = 1$, $K = 500$, $D = 3$, $c_1 = \frac{1}{3}$

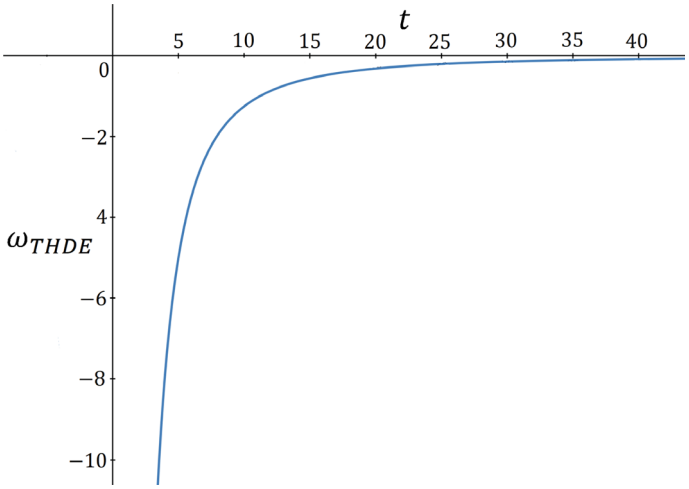

Fig. 8 The plot of ω_{THDE} versus cosmic time t graph with $\rho_0 = 3, l = 1, K = 500, \alpha = -2, D = 3$

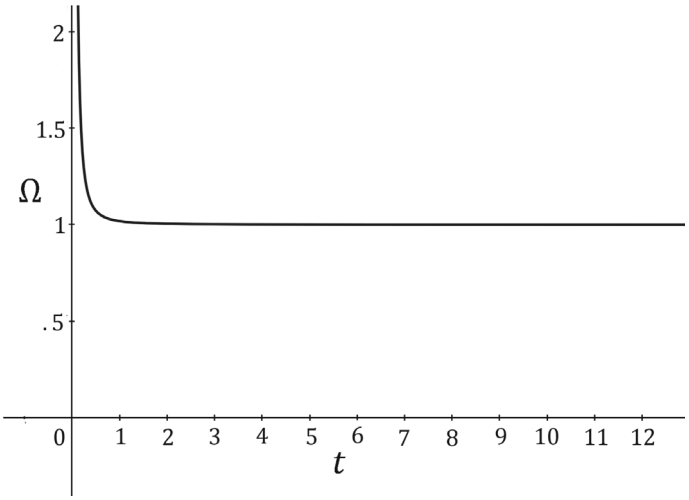

Fig. 9 The plot of total energy density Ω versus cosmic time t graph with $\rho_0 = 3, l = 1, K = 500, \alpha = -2, D = 3$

where Q is the coupling term between THDE and the cold dark matter. Following the work of Wei and Cai [28], we choose

$$Q = 3\alpha H \rho_m \tag{55}$$

where α is the coupling constant.

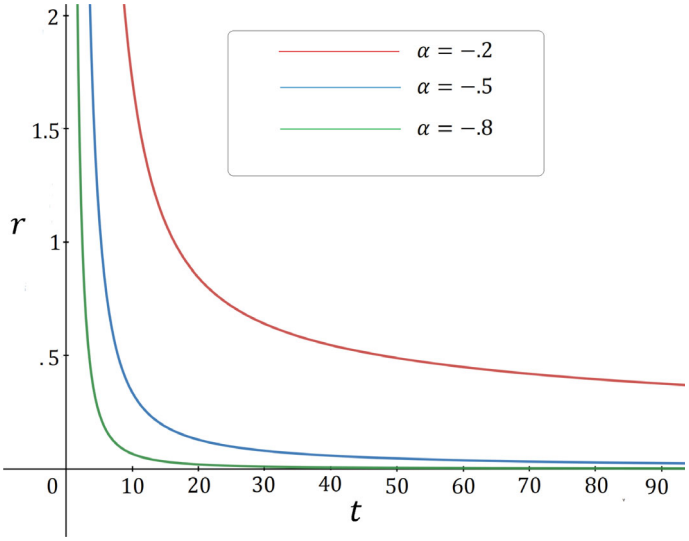

Fig. 10 The plot of cosmic coincidence parameter r versus cosmic time t graph with $\rho_0 = 3, l = 1, K = 500, D = 3$

Then from (38), (53) and (55), we obtain

$$\rho_m = \rho_0 (t^2 + \frac{K}{4l^2})^{(\alpha-1)} \tag{56}$$

where ρ_0 is an integrating constant.

For $\alpha < 1$, ρ_m is positive for all t and decreases to 0 as $t \to \infty$.

Therefore the EoS parameter is given by

$$\omega_{THDE} = \frac{p_{THDE}}{\rho_{THDE}} = -1 - \frac{1}{3H}\frac{\dot{\rho}_{THDE}}{\rho_{THDE}} - \alpha\frac{\rho_m}{\rho_{THDE}} = -1 + (-2\delta + 4)\left(1 - \frac{t^2 + \frac{K}{4l^2}}{2t^2}\right) - \alpha\frac{\rho_0 (t^2 + \frac{K}{4l^2})^{(\alpha-1)}}{D\left[\frac{2t}{3(t^2+\frac{K}{4l^2})}\right]^{-2\delta+4}} \tag{57}$$

In this case also, the universe contains phantom energy at early stage of the universe and non-phantom energy at late times.

The density parameters Ω_m and Ω_{THDE} are obtained as

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{3\rho_0}{4t^2}(t^2 + \frac{K}{4l^2})^{\alpha+1} \tag{58}$$

$$\Omega_{THDE} = \frac{\rho_{THDE}}{3H^2} = \frac{3D(t^2 + \frac{K}{4l^2})^2}{4t^2}.[\frac{2t}{3(t^2 + \frac{K}{4l^2})}]^{-2\delta+4} \tag{59}$$

Hence,

$$\Omega = \Omega_m + \Omega_{THDE} = \frac{\rho_m}{3H^2} = \frac{3\rho_0}{4t^2}(t^2 + \frac{K}{4l^2})^{\alpha+1} + \frac{3D(t^2 + \frac{K}{4l^2})^2}{4t^2}.\left[\frac{2t}{3(t^2 + \frac{K}{4l^2})}\right]^{-2\delta+4} \tag{60}$$

The total energy density tends to 1 at late time; hence the universe is close to flat. The cosmic coincidence parameter is given by

$$r = \frac{\rho_m}{\rho_{THDE}} = \frac{\rho_0\left(t^2 + \frac{K}{4l^2}\right)^{(\alpha-1)}}{D\left[\frac{2t}{3(t^2+\frac{K}{4l^2}}\right]^{-2\delta+4}} \tag{61}$$

From the graph and the Eq. (61), it can be shown that at $t \to \infty$, $r = a$ constant for $-1 < \alpha < 0$. For $\alpha \leq -1$, $r \to 0$.

Hence, the cosmic coincidence problem can be resolved at late time for $-1 < \alpha < 0$. And for $0 < \alpha < 1$, r is divergent.

4 Conclusion

In this paper we study Tsallis Holographic dark energy (THDE) model in LRS Bianchi type II universe within the framework of General Relativity. We consider the energy density ρ_{THDE} of THDE with the identification of IR cut-off as Hubble horizon given by $\rho_{THDE} = DH^{-2\delta+4}$, D being an unknown parameter. Exact solutions of Einstein's field equations are obtained for two cases depending on a constant of integration $C = 0$ and $C \neq 0$ by connecting the directional scale factors A and B through a relation $A = lB^m$, where l and m are positive constants. As $C = 0$ does not lead to accelerated expansion for all t, so this model represents the matter dominated era of the universe and we do not discuss much about it. For $C \neq 0$, we consider both non-interacting and interacting THDE models for the particular value of $m = \frac{2}{3}$. For $C > 0$, the solution does not exhibit accelerated expansion of the universe for any value of the cosmic time t. For $C < 0$, the universe starts at $t = \frac{\sqrt{K}}{2l}$ and expands

with acceleration till $t < \frac{\sqrt{3K}}{2l}$, where $K = -C$. After that the universe expands with deceleration till the end.

The model we considered in case I is a deceleration model of the universe. But in that case, the universe starts with an initial big bang at $t = 0$ with volume $V = 0$ and goes on expanding with deceleration. The model we considered in the second case is the present accelerating model of the universe, but in this case we also get a deceleration of the universe in the future. In this case, the universe does not start at $t = 0$ with volume $V = 0$. So, we cannot explain the initial big bang singularity of the universe in this case. The second model is a very good model of the early inflationary and present-accelerating universe.

The Tsallis holographic dark energy density is maximum at the start of the universe and decreases as the universe evolves. For $t \geq \frac{\sqrt{3K}}{2l}$, though the universe expands with deceleration but A, B, V are infinite at late time and $H \to 0$ as $t \to \infty$. Hence, our model reaches Big Rip in future. For both the cases, interacting and non-interacting, the universe approaches flat one at late time. Also the universe remains anisotropic throughout its evolution.

For the non-interacting THDE model, the universe contains phantom energy at early stage and non-phantom energy after the evolution. For the phase of accelerated expansion, the THDE behaves like quintessence scalar field. The cosmic coincidence parameter r becomes constant for $1 \leq \delta < 2$ as $t \to \infty$, and hence, the cosmic coincidence problem can be resolved at late time. For the interacting THDE model with the coupling parameter $Q = 3\alpha H\rho_m$, the matter energy density ρ_m is positive for all t and decreases to 0 as $t \to \infty$ for $\alpha < 0$. Also, for $\alpha < 0$, the universe contains phantom energy at early stage and non-phantom energy after the evolution. The coincidence problem can also be resolved for $-1 < \alpha < 0$.

For the non-interacting THDE model, the coincidence parameter asymptotically tends to zero if we consider δ relatively large. For the interacting THDE model, the coincidence parameter asymptotically tends to zero if we consider α relatively smaller.

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